Measuring ambiguity attitude: (extended) multiplier preferences for the American and the Dutch population

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ABSTRACT: Empirical measurements of ambiguity aversion typically use measures that are not founded in theory. This paper shows how a theoretically-founded measure of ambiguity aversion can be derived from Hansen and Sargent's theory of multiplier preferences. Multiplier or robust preferences are widely used in macroeconomics to capture model uncertainty. At the micro level, they have not been applied yet, because they do not permit ambiguity seeking, which is usually observed for a substantial proportion of subjects. We give a preference foundation for (extended) multiplier preferences that allows for both ambiguity aversion and ambiguity seeking. We then propose a simple method to measure multiplier preferences, which gives an axiomatically founded measure of ambiguity attitude. We illustrate our method in two large representative samples (one Dutch and one American) and obtain the first micro estimates of multiplier preferences. Nearly one third of the respondents are ambiguity seeking, illustrating the need for extended multiplier preferences.

KEY WORDS: ambiguity, multiplier preferences, robustness, measurement.

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1. Introduction

While both the theoretical and the empirical literature on ambiguity are rich,[1] there is only limited interaction between the two. A reason is that most ambiguity models use concepts that are hard if not impossible to observe empirically. Empirical measurements of ambiguity have therefore resorted to pragmatic measures that lack a foundation in theory. The purpose of this paper is to bridge this gap between theory and empirics. We use Hansen and Sargent's (2001) *multiplier preferences* model, which captures ambiguity aversion by a single parameter, to derive a theoretically-founded measure of ambiguity aversion. We extend the multiplier preference model to capture all kinds of ambiguity attitudes, we present a method to measure the ambiguity aversion parameter, and we apply this method in two large representative surveys.

Multiplier preferences are widely used in macroeconomics and finance to permit that decision makers' beliefs about economic phenomena are non-unique. In the multiplier preferences model, decision makers rank payoff profiles f according to the criterion:

[1] See Trautman and van de Kuilen (forthcoming) for a recent survey of the empirical literature and Machina and Siniscalchi (2014) for a survey of the theoretical literature.

$$V(f) = \min_p \int u(f)\, dp + \frac{1}{\sigma} R(p||q), \tag{1}$$

where u is a utility function, q is a subjective probability distribution on the states of the world, σ is a behavioral parameter, and $R(p||q)$ is the relative entropy of any probability distribution p with respect to q. The intuition underlying Eq. (1) is that the decision maker has some best guess q of the probability distribution, but he does not have full confidence in his guess and also considers other probability distributions p. The plausibility of these other distributions decreases with their distance from q, as measured by the relative entropy R. The parameter $\frac{1}{\sigma}$ captures the degree to which the decision maker takes alternative probability distributions into account. The lower is σ, the more the decision maker trusts that q is the correct distribution. In the limit, if σ goes to zero, Eq. (1) becomes subjective expected utility.

The lack of trust decision makers have in their beliefs may result from ambiguity (Hansen and Sargent 2001). In empirical studies, most subjects are not neutral towards ambiguity, as assumed by expected utility, but are ambiguity averse. Multiplier preferences capture ambiguity aversion while remaining analytically convenient and easy to incorporate in economic models of aggregate behavior. However, they do not accommodate ambiguity seeking, which limits their applicability at the micro level where a wide range of ambiguity attitudes is typically observed and a substantial proportion of respondents is ambiguity seeking.

This paper extends multiplier preferences to accommodate both ambiguity aversion and ambiguity seeking. We give a preference foundation of this extended model that complements Strzalecki (2011) and that makes multiplier preferences suitable for microeconomic applications.

We then present a simple method to measure extended multiplier preferences. Our method is easy to apply and measures multiplier preferences at the individual subject level. Hence, we obtain an axiomatically founded measure of ambiguity aversion that can easily be used in empirical research and that captures the heterogeneity in individual ambiguity attitudes.

We illustrate our method in two large representative samples of the Dutch and the US population involving over 5,000 subjects in total and provide the first micro estimates of (extended) multiplier preferences. Most subjects were moderately ambiguity averse, but between 23% (Dutch sample) and 36% (US sample) were ambiguity seeking. In both samples, we observed that education and income were uncorrelated with ambiguity aversion but negatively correlated with the deviation from ambiguity neutrality.

2. Extended multiplier preferences

We use the Anscombe-Aumann setting. Let S be the *state space,* i.e. the set of all possible *states of the world* s. S can be finite or infinite. One state s will occur but the decision maker does not know which one. Σ denotes a sigma-algebra on S. Its elements are called *events* and are typically denoted E. The set of all countably additive probability measures on (S, Σ) is denoted by $\Delta(S)$ and is endowed with the weak* topology. A probability measure $p \in \Delta(S)$ is *absolutely continuous* with respect to $q \in \Delta(S)$ if for all $E \in \Sigma$, $q(E) = 0$ implies $p(E) = 0$. Let $\Delta(q)$ denote the set of all countably additive probability measures that are absolutely continuous with respect to q. For any $p, q \in \Delta(S)$, the *relative entropy* of p with respect to q is given by $R(p||q) = \int_S \log\left(\frac{dp}{dq}\right) dp$ if $p \in \Delta(q)$ and $R(p||q) = \infty$ otherwise.

We denote the *outcome set* by Z. $\Delta(Z)$ is the set of all simple lotteries on Z. Elements of $\Delta(Z)$ are denoted x or y. The decision maker chooses between *acts*, finite-valued mappings from S to $\Delta(Z)$, which are Σ-measurable. Acts are usually denoted f or g. For event E, $f_E g$ denotes the act that gives $f(s)$ if $s \in E$ and $g(s)$ if $s \in E^c$ with E^c the complement of E. The set of all acts is $\mathcal{F}$. Acts have two stages: the first stage corresponds to the uncertainty modeled by S and the second stage to the risks modeled by $\Delta(Z)$. The *mixture act* $\alpha f + (1-\alpha)g$ for $\alpha \in [0,1]$ is the act that assigns the lottery $\alpha f(s) + (1-\alpha)g(s)$ to state s for all $s \in S$. The decision maker's preferences over acts in $\mathcal{F}$ are denoted by $\succcurlyeq$ (with $\sim$, $\succ$, $\preccurlyeq$, and $\prec$ defined as usual). A functional V *represents* $\succcurlyeq$ if $V: \mathcal{F} \to \mathbb{R}$ is such that $f \succcurlyeq g \Leftrightarrow V(f) \geq V(g)$.

Definition 1: We call $\succcurlyeq$ *extended multiplier preferences* if $\succcurlyeq$ can be represented by

$$V(f) = \begin{cases} \min_{p \in \Delta(S)} \int_S u(f(s))dp(s) + \frac{1}{\sigma}R(p||q) & if\ \sigma > 0 \\ \int_S u(f(s))dp(s) & if\ \sigma = 0 \\ \max_{p \in \Delta(S)} \int_S u(f(s))dp(s) + \frac{1}{\sigma}R(p||q) & if\ \sigma < 0 \end{cases}$$

where u is a nonconstant expected utility functional, $q \in \Delta(S)$, and $\sigma \in \mathbb{R}$. We call these preferences robust if $\sigma \geq 0$ and opportunity seeking if $\sigma \leq 0$.

A decision maker whose preferences are opportunity seeking chooses the probabilities that will maximize his expected utility minus a cost, which depends on the distance between these probabilities and his best guess. A decision maker with robust preferences tries to find options that are maximally insensitive to remaining uncertainties. By contrast, an opportunity seeking decision maker is looking for possibilities to improve his expected utility and he values options for which the remaining uncertainties can lead to high expected utilities.

An alternative interpretation of extended multiplier preferences approach comes from a comparison with $\int u(f)\, dp + \theta[R(p||q) - \eta]$, the Lagrange function deduced from minimizing (in the robust approach) or maximizing (in the opportunity seeking approach) $\int u(f)\, dp$ such that the relative entropy does not exceed a threshold ($R(p||q) < \eta$). This comparison shows that the multiplier parameter $\theta = \frac{1}{\sigma}$ is the Lagrange multiplier of the optimization problem and can be interpreted as the shadow price of relaxing the constraint imposed on the relative entropy (Hansen and Sargent, 2001).

There is a third interpretation of the multiplier parameter as an index of ambiguity aversion. Lemma A1 in the Appendix shows that extended multiplier preferences are ordinally equivalent to second-order expected utility[2] (SOEU)

$$V(f) = \int_S \varphi_\sigma \left(u\big(f(s)\big)\right) dq(s)$$

when φ_σ is exponential:

$$\varphi_\sigma(t) = \begin{cases} -e^{-\sigma t} & if\ \sigma > 0 \\ t & if\ \sigma = 0 \\ e^{-\sigma t} & if\ \sigma < 0 \end{cases}$$

and u, q, and σ as in Definition 1. Axiomatizations of SOEU were given by Grant, Polak, and Strzalecki (2009), Nau (2006), and Neilson (2010). We know from Pratt (1964) that under expected utility the exponential utility function is equivalent to constant absolute risk aversion. This implies that adding an amount c to all outcomes of the lotteries under comparison does not change the preferences

[2] SOEU shows that ambiguity attitudes can be modeled by relaxing the assumption of reduction of compound lotteries between the objective stage (the lottery $f(s)$) and the subjective stage (the subjective probability $q(s)$). Segal (1987) first made this point using rank-dependent utility in both stages. Dillenberger and Segal (2015) showed that Segal's model also accommodates examples of ambiguity behavior proposed by Machina (2009, 2014) that most other ambiguity models cannot accomodate.

between these lotteries. For the exponential function, the Arrow-Pratt index of risk attitude $-\frac{u''}{u'}$ is constant and equal to the exponential parameter. Under SOEU, we can give a similar interpretation to the exponential φ_σ function in terms of utility: adding the same (expected) utility to each state of the acts under comparison does not change the preferences between these acts. Grant and Polak (2013) coin the term constant absolute uncertainty aversion to describe this property. The index $-\frac{\varphi''}{\varphi'} = \sigma$ is then an Arrow-Pratt index of ambiguity attitude. Hansen and Sargent (2001) used $\theta = \frac{1}{\sigma}$ as an ambiguity measure. We used σ instead of θ, because σ is a monotonic and continuous measure and, therefore, more convenient for statistical analysis.

3. Axiomatization

Strzalecki (2011) axiomatized extended multiplier preferences for $\sigma \geq 0$, i.e. for decision makers with robust preferences. We will characterize extended multiplier preferences, i.e. including the case of opportunity seeking ($\sigma \leq 0$). We do so by dropping uncertainty aversion (his A.5) from Strzalecki's set of axioms and by replacing results in his proof that depend on this axiom by other results that do not depend on it.

We impose the following conditions on $\succcurlyeq$:

1. *Weak order:* $\succcurlyeq$ is complete and transitive.
2. *Weak certainty independence:* for all $f, g \in \mathcal{F}$, for all $x, y \in \Delta(Z)$, and for all $\alpha \in (0,1)$, $\alpha f + (1-\alpha)x \succcurlyeq \alpha g + (1-\alpha)x \Rightarrow \alpha f + (1-\alpha)\mathrm{y} \succcurlyeq \alpha g + (1-\alpha)y$.
3. *Continuity:* for all $f, g, h \in, \mathcal{F}$, the sets $\{\alpha \in [0,1]: \alpha f + (1-\alpha)\mathrm{g} \succcurlyeq h\}$ and $\{\alpha \in [0,1]: \alpha f + (1-\alpha)\mathrm{g} \preccurlyeq h\}$ are closed.
4. *Monotonicity:* for all $f, g \in \mathcal{F}$ if $f(s) \succcurlyeq g(s)$ for all $s \in S$ then $f \succcurlyeq g$.
5. *Nondegeneracy*: there exist acts $f, g \in \mathcal{F}$ such that $f \succ g$.
6. *Weak monotone continuity:* for all $f, g \in \mathcal{F}$, for all $x \in \Delta(Z)$, and for all $\{E_n\}_{n\geq 1} \in \Sigma$ with $E_1 \supseteq E_2$ …. and $\cap_{n\geq 1} E_n = \emptyset$, $f \succ g$ implies that there exists an n_0 such that $x_{E_{n_0}} f \succ g$.
7. *Sure thing principle:* for all $E \in \Sigma$ and for all $f, g, h, h' \in \mathcal{F}$, $f_E h \succcurlyeq g_E h \Rightarrow f_E h' \succcurlyeq g_E h'$.

An event is *essential* if there exist $f, g, h \in \mathcal{F}$ such that $f_E h \succ g_E h$.

Theorem 1: If S has at least three disjoint essential events[3] then the following two statements are equivalent:

1. $\succcurlyeq$ is a continuous, nondegenerate weak order that satisfies weak certainty independence, monotonicity, weak monotone continuity and the sure thing principle.
2. $\succcurlyeq$ has an extended multiplier representation.

Observation 1: Two triples (σ, u, q) and (σ', u', q') represent the same extended multiplier preference if and only if q and q' are identical and there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u' = \alpha u + \beta$ and $\sigma' = \sigma/\alpha$.

All proofs are in the Appendix.

We can distinguish the robust and the opportunity seeking approaches using Schmeidler's (1989) condition of ambiguity aversion and its counterpart of ambiguity seeking.

[3] If only one event is essential then the Theorem also holds but the uniqueness properties are different. If exactly two disjoint events are essential then the sure thing principle should be strengthened to the hexagon condition (Wakker 1989).

Definition 2: *Ambiguity aversion* (*seeking*) holds if for all acts f, g in $\mathcal{F}$ and for all α in $(0,1)$, $f \sim g \Longrightarrow \alpha f + (1-\alpha) g \succcurlyeq (\preccurlyeq) f$.

Theorem 2: Under extended multiplier preferences, ambiguity aversion is equivalent to robust preferences and ambiguity seeking is equivalent to opportunity seeking preferences.

According to Theorem 2, the sign of σ determines whether an agent is ambiguity averse or ambiguity seeking. But for σ to be a proper index of ambiguity aversion, it should also satisfy the property that a higher value represents more ambiguity aversion. Consider two decision makers $i \in \{1,2\}$ represented by preferences $\succcurlyeq_i$. We use the definition of "more ambiguity averse" proposed by Ghirardato and Marinacci (2002).

Definition 3: $\succcurlyeq_2$ is *more ambiguity averse than* $\succcurlyeq_1$ if for all acts f in $\mathcal{F}$ and lotteries x in $\Delta(Z)$, $x \succcurlyeq_1 f \Longrightarrow x \succcurlyeq_2 f$.

This definition adapts the definition of "more risk averse" introduced by Yaari (1969) to ambiguity. It implies that the ambiguity attitudes of two decision makers can only be compared if they share the same beliefs (here, the same q). Moreover, as shown by Ghirardato and Marinacci (2002, Proposition 11), the decision makers need to have the same risk attitudes, which corresponds to their utility functions being cardinally equivalent: $u_1 \approx u_2$ if there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u_1 = \alpha u_2 + \beta$.

Theorem 3: Given two extended multiplier preferences $\succcurlyeq_1$ and $\succcurlyeq_2$ represented by (σ_1, u_1, q_1) and (σ_2, u_2, q_2), the following two statements are equivalent:

1. $\succcurlyeq_2$ is more ambiguity averse than $\succcurlyeq_1$.
2. $u_1 \approx u_2$, $q_1 = q_2$, and $\sigma_1 \leq \sigma_2$ (if we scale utility such that $u_1 = u_2$).

Theorem 3 shows that σ is a proper measure of ambiguity aversion.

4. Measuring extended multiplier preferences

4.1. Method

Strzalecki (2011, Example 3) explained how the multiplier parameter σ could be measured under the assumption that utility u is a power function. We describe an alternative method that makes no assumptions about utility and requires fewer questions. Because extended multiplier preferences are ordinally equivalent to SOEU with φ_σ exponential, we will display our results using SOEU for ease of understanding. Suppose that a ball will be drawn from an urn with an unknown number of yellow and purple balls. Let $S = \{Y, P\}$ where Y stands for "the ball is yellow" and P for "the ball is purple". The decision maker can win either \$15 or nothing, depending on the color of the ball. Hence, $Z = \{0{,}15\}$. The act f_Y pays \$15 if the ball is yellow and nothing otherwise and the act f_P pays \$15 if the ball is purple and nothing otherwise. Each lottery from $\Delta(Z)$ can be written as 15_r0, where r is the probability to get 15. We scale utility so that $u(0) = 0$ and $u(15) = 15$. Then $u(15_r0) = r * 15 + (1 - r) * 0 = 15r$.

Assume $f_Y \sim f_P \sim 15_r0$ for some probability r. We call this probability r a *matching probability* of the acts f_Y and f_P. Under SOEU, we obtain $q(Y) = q(P) = ½$ from $f_Y \sim f_P$. The second indifference, $f_P \sim 15_r0$, then implies $\varphi_\sigma(15r) = ½\, \varphi_\sigma(15) + ½\, \varphi_\sigma(0)$. We prove in the Appendix that this equation has a unique solution σ for each value of $r \in (0{,}1)$. If $r = ½$, then $\sigma = 0$ and the decision maker is indifferent between an objective and a subjective probability of ½ . If $r < ½$ then $\sigma > 0$ and the decision maker prefers an objective probability of ½ to a subjective probability of ½. This corresponds to ambiguity aversion. Similarly, $r > ½$ implies ambiguity seeking ($\sigma < 0$). If $r \to 0$, preferences are extremely robust (ambiguity averse) and $\sigma \to +\infty$. If $r \to 1$, preferences are extremely opportunity seeking and $\sigma \to -\infty$.

4.2. Calibration

Observation 1 shows that the sign of the multiplier parameter does not depend on the scaling of the utility function, but its magnitude does. In the empirical study reported in Section 5, we scale utility such that the utility of initial wealth W is 0 and that of $W + 15$ is 15. For any utility function v, the corresponding multiplier parameter σ_v can be computed from the σ that we report below using $\sigma_v = \frac{15}{v(W+15)-v(W)}\sigma$. Because σ depends only on the scaling of utility and not on utility curvature, it does not depend on a subject's risk aversion. Hence, it can be used for correlation analysis if the same scaling is used for all subjects.

4.3. Empirical illustration

Two surveys have been held in which subjects answered questions of the form described in section 4.1. Dimmock, Kouwenberg and Wakker (2015) ran a survey among 1,900 participants of the Dutch Longitudinal Internet Study for the Social Sciences (LISS). Dimmock, Kouwenberg, Mitchell and Peijnenburg (2013) ran a similar survey among 3,300 participants of the American Life Panel (ALP)[4]. We

[4] Both papers analyzed a subset of their respondents, excluding subjects who took too much or too little time in answering., For example. Dimmock et al. (2015) excluded more than half of their subjects as these were not incentivized. In our analyses, we chose to include all subjects as any

illustrate our method by showing the σ values obtain from the responses in these two datasets.

In both surveys, subjects had to choose between two urns: a known urn K and an ambiguous urn A. Urn K contained 100 yellow (orange) and purple balls in known proportions. Urn A contained 100 yellow and purple balls in unknown proportions. By default, purple was the winning color, but subjects could change the winning color in the Dutch survey. Only 1% of all subjects did so, which indicates that most subjects were not suspicious and had no preference between the two winning colors. This implies $f_Y \sim f_P$.

The survey measured the matching probability r for which subjects were indifferent between urn A and urn K with $r * 100$ balls of their winning color. Subjects made a series of choices between urn A and urn K, where urn A remained the same while the proportion of winning balls in urn K changed depending on previous choices.

At the end of the experiments, one randomly selected choice was played for real. A ball was drawn from the urn that the subject preferred in that choice. The subject received 15 euro (dollar) if the ball had his winning color and nothing otherwise.

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exclusion criterion is to some extent arbitrary. The results we present were unaffected if we used the same inclusion criteria as Dimmock et al. (2013, 2015).

Advantages and disadvantages of the approached followed are discussed in Dimmock et al. (2013, 2015).

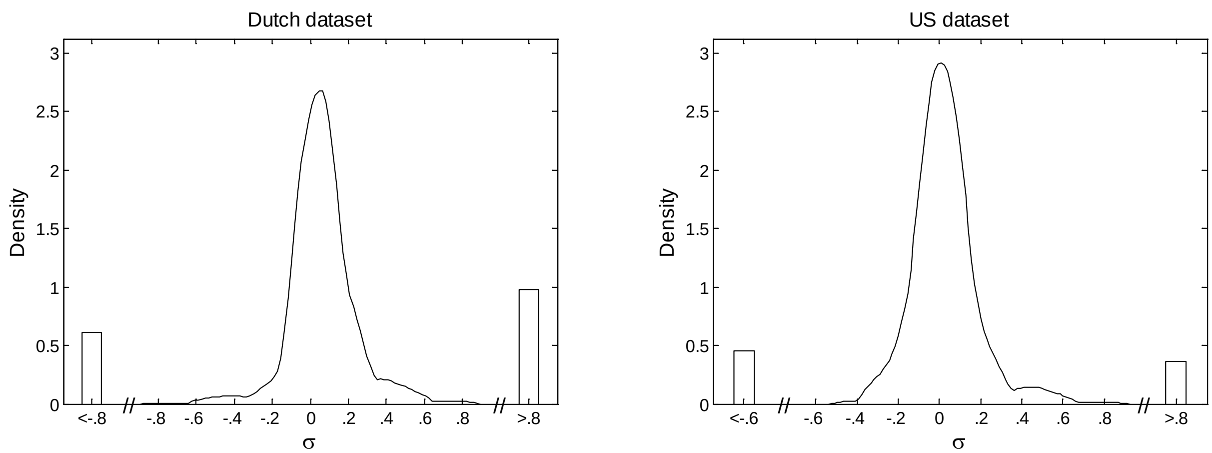

Figure 1: Kernel density estimates of respondents' σ values. The Epanechnikov function was used, with a kernel width of 0.07. The boxes at the upper and lower end indicate the proportion of subjects with σ values greater than .8 and less than -.8/-.6.

Figure 1 shows the estimated distribution of σ in the two datasets using a kernel density estimate. In the Dutch (US) dataset, the median value of σ was equal to 0.05 (0.02)), which corresponds with a matching probability of 40.6% (47.0%). Both distributions are centered slightly to the right of zero and concentrated in the ambiguity averse domain. Still, 22.5% (35.9%) of subjects were found to be ambiguity seeking. The box at the far left of the distribution show that 6.2% (4.5%) of the subjects gave matching probabilities close to 1, which corresponds with a value of σ that is below -0.8 (-0.6)[5]. Similarly, the boxes on the far right indicate that 9.6% (3.6%) gave matching probabilities close to zero, which corresponds with a σ value greater than 0.8.

	Dutch dataset		US dataset	
	σ	$\lvert\sigma\rvert$	σ	$\lvert\sigma\rvert$
Gender (female = 1)	0.012	−0.002	−0.053***	0.011
Age	−0.070***	0.142***	−0.032*	0.001
High income	−0.003	−0.047**	0.020	−0.060***
High education	0.003	−0.064***	0.031*	−0.064***
N	1,821	1,821	3,217	3,217

Table 1: Correlations between demographic variables and ambiguity aversion (σ) and deviation from ambiguity neutrality ($|\sigma|$). *significant at 10% level, **5%, ***1%.

As a further illustration, Table 1 shows correlations between σ and demographic variables in the first and third column, and correlations between $|\sigma|$ and demographic variables in the second and fourth column. Correlations with $|\sigma|$ are also analyzed because some effects may be correlated with the deviation from ambiguity neutrality, rather than with the degree of ambiguity aversion. Such a deviation implies a violation of either probabilistic sophistication or dynamic

[5] These thresholds are not of the same absolute value due to an asymmetry in the question design of Dimmock et al. (2013).

consistency, two conditions that are generally considered normative. Ambiguity neutrality is therefore often perceived as the rational model of choice under uncertainty (e.g., Wakker 2010, p. 326).

In the Dutch sample, the only variable that is correlated with σ is age, with older respondents being more ambiguity seeking. The second column shows that age is *positively* correlated with $|\sigma|$, which suggests that they have more extreme ambiguity attitudes. Income and education are negatively correlated with the deviation from ambiguity neutrality, which seems consistent with the finding that people with higher cognitive abilities deviate less from models of rational choice (Frederick 2005, Dohmen et al. 2010).

In the US sample, women are more ambiguity seeking, as are older and less educated people (marginally significant). Although there is no correlation between age and $|\sigma|$ as in the Dutch dataset, the correlation coefficients for income and education are remarkably similar to their Dutch counterparts. All correlations are negative, indicating that those with higher income and education are closer to ambiguity neutrality.

5. Concluding remarks

Multiplier preferences, proposed by Hansen and Sargent (2001), are a popular model in macroeconomics and finance. In its original form, multiplier preferences only capture ambiguity aversion, which make them less suitable for applications at the micro level where substantial ambiguity seeking has also been observed. This paper extends multiplier preferences to include ambiguity seeking and it gives a preference foundation for these extended multiplier preferences. We also show how extended multiplier preferences can be measured and thereby obtain an axiomatically-founded measure of ambiguity aversion that can easily be applied in empirical studies and that captures the substantial heterogeneity in ambiguity attitudes that typically exists in micro data. As an illustration, we applied our method to two large scale representative surveys, one from the Netherlands and one from the US. In both samples a substantial fraction of the respondents was ambiguity seeking, which illustrates the desirability of our extension of multiplier preferences.

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Appendix

Lemma A1: Preferences $\succcurlyeq$ are extended multiplier preferences if and only if there exists $\sigma \in \mathbb{R}$ such that $\succcurlyeq$ can be represented by SOEU with $q \in \Delta(S)$ and $\varphi = \varphi_\sigma$.

Proof:

The equivalence between robust preferences and $\varphi(t) = -e^{-\sigma t}$ has been shown by Strzalecki (2011). It is based on Proposition 1.4.2 of Dupuis and Ellis (1997) stating that for all countably additive probability measures $q \in \Delta(S)$ and for all Σ-measurable functions v:

$$\min_{p \in \Delta(S)} \int_S v(s) dp(s) + \frac{1}{\lambda} R(p||q) = \varphi_\lambda^{-1}\left(\int_S \varphi_\lambda\big(v(s)\big) dq(s)\right).$$

For $\sigma < 0$, we apply this formula to $v = -u \circ f$ and $\lambda = -\sigma$ and we obtain:

$$\begin{aligned} \max_{p \in \Delta(S)} \int_S u\big(f(s)\big) dp(s) + \frac{1}{\sigma} R(p||q) &= -\left[\min_{p \in \Delta(S)} \int_S v(s) dp(s) + \frac{1}{\lambda} R(p||q)\right] \\ &= -\varphi_\lambda^{-1}\left(\int_S \varphi_\lambda\big(v(s)\big) dq(s)\right) \\ &= \varphi_\sigma^{-1}\left(\int_S \varphi_\sigma\Big(u\big(f(s)\big)\Big) dq(s)\right). \end{aligned}$$

The last equality follows from $\varphi_\sigma^{-1}(t) = -\frac{\ln(t)}{\sigma} = \frac{\ln(t)}{\lambda} = -\varphi_\lambda^{-1}(-t)$ and

$\varphi_\lambda\big(v(s)\big) = -e^{-\lambda v(s)} = -e^{-\sigma u(f(s)))} = -\varphi_\sigma\Big(u\big(f(s)\big)\Big)$.

Hence, both robust and opportunity seeking preferences are equivalent to SOEU with an exponential φ function. ■

Proof of Theorem 1:

(ii) $\Rightarrow$ (i). Because (ii) is a normalized niveloid that represents $\succcurlyeq$ and u is nonconstant and affine, Lemma 28 in Maccheroni, Marinacci and Rustichini (2006) implies that $\succcurlyeq$ is a constinuous, nondegenerate weak order that satisfies weak certainty independence and monotonicity. Because q is countably additive, $\succcurlyeq$ satisfies uniform continuity by Theorem 5.4 in Krantz et al. (1971). Finally, by Proposition 1.4.2 in Dupuis and Ellis (1997), (ii) is equivalent to a second order expected utility representation. Consequently, the sure thing principle must hold.

We show that (i) $\Rightarrow$ (ii) by closely following Strzalecki's proof without imposing uncertainty aversion. First we introduce some new notation. Let $B_0(\Sigma)$ denote the set of all real-valued Σ-measurable simple functions[6] and let $B_0(\Sigma, \mathrm{K})$ denote the set of functions in $B_0(\Sigma)$ that take values in a convex set $K \subseteq \mathbb{R}$. Let $\Phi_3$denote the set of finite partitions of S that contain at least three essential events. For all $G \in \Phi_3$, let $\mathcal{A}(G)$ be the algebra generated by G and let $\mathcal{F}_G$ denote the set of acts in $\mathcal{F}$ that are measurable with respect to $\mathcal{A}(G)$.

By Lemmas 25 and 28 of Maccheroni et al. , there exist a real-valued nonconstant affine function u on $\Delta(Z)$ and a normalized real-valued functional $I: B_0(\Sigma, \mathcal{U}) \to \mathbb{R}$ where $\mathcal{U}$ is the range of $u\big(\Delta(Z)\big)$ and such that for all acts $f, g \in \mathcal{F}$, $f \succcurlyeq g$ iff

[6] A function is simple if it takes no more than countably many distinct values.

$I(u \circ f) \geq I(u \circ g)$ and $I(\alpha\psi + (1-\alpha)k) = I(\alpha\psi) + (1-\alpha)k$ for all $\psi \in B_0(\Sigma, \mathcal{U})$, $k \in \mathcal{U}$ and $\alpha \in (0,1)$.

Theorem 1 in Grant, Polak, and Strzalecki (2009) ensures that for finite $S \succcurlyeq$ can be represented by $f \longmapsto \sum_{s\in S} v_s(u(f(s)))$ with u nonconstant and affine and with range $\mathcal{U}$ and v_s continuous, nondecreasing, and with at least three v_s nonconstant. Weak certainty independence then ensures that indifference curves in the utility space are parallel and have common supporting hyperplanes at the set of constant vectors in $\mathcal{U}^S$. By the proof of Theorem 3 in Grant et al. (2009) it follows that for all $G \in \Phi_3$the restriction of $\succcurlyeq$ to $\mathcal{F}_G$ can be represented by $f \longmapsto \sum_{s\in S} p_G(s)\varphi_G\big(u_G(f_s)\big)$ with u_G nonconstant and affine, φ_G continuous and strictly increasing, and measure $p_G\colon \mathcal{A}(\mathrm{G}) \to [0,1]$ such that at least three events in G are nonzero. In applying Theorem 3, we replace uncertainty aversion and their Axiom A.7 by weak certainty independence. Uncertainty aversion is used in the application of Theorem 3 in Debreu and Koopmans (1982) to derive differentiability of the functions v_s. However, as noted by Grant et al. (2009) and Maccheroni et al. (p.1475, 1491), weak certainty independence implies Lipschitz continuity and hence absolute continuity of the v_s functions so that they can be represented as integrals of their (almost everywhere) derivatives. By Theorem 4 in Strzalecki (2011), the proof of which does not use uncertainty aversion, $\succcurlyeq$ can be represented by second order expected utility $f \longmapsto \int_S \varphi\big(u(f_s)\big)dq(s)$ with $q \in \Delta(Z)$ and φ continuous and strictly increasing. q is countably additive by uniform continuity (Villegas 1964, Theorem

1). Moreover, if (u, φ, q) and (u', φ', q') both represent $\succcurlyeq$ then there exist $\alpha, A > 0, \beta, B \in \mathbb{R}$ such that $q^{'} = q, u^{'} = \alpha u + \beta, \varphi^{'}(\alpha r + \beta) = A\varphi(r) + B$ for all r in $\mathcal{U}$.

I represents $\succcurlyeq$ and is translation invariant, i.e. for all $f, g \in \mathcal{F}$ and k such that $f(s) + k, g(s) + k \in \mathcal{U}$ for all $s \in S$, $I(u \circ f) \geq I(u \circ g)$ iff $I(u \circ f + k) = I(u \circ f) + k \geq I(u \circ g) + k = I(u \circ g + k)$. It then follows that for all acts $f, g \in \mathcal{F}$ and k such that $f(s) + k, g(s) + k \in \mathcal{U}$ for all $s \in S$, $\int_S \varphi(u(f(s)))dq(s) \geq \int_S \varphi(u(g(s)))dq(s)$ iff $\int_S \varphi(u(f(s) + k)dq(s) \geq \int_S \varphi(u(g(s) + k)dq(s)$.

Hence, (u, φ, q) and (u, φ_k, q) defined by $\varphi_k(l) = \varphi(l + k)\ \forall l, l + k \in \mathcal{U}$ are both SOEU representations of $\succcurlyeq$. Consequently, $\varphi(l + k) = A(k)\varphi(l) + B(k)$. Because φ is nonconstant, if $\mathcal{U}$ is unbounded, it follows from Corollary 1 in Aczél (1966, Section 3.1.3) that φ equals φ_σ. If $\mathcal{U}$ is bounded then because φ is nonconstant Theorem 4 in Aczél (2005) implies $\varphi = \varphi_\sigma$ on the interior of $\mathcal{U}$. Because φ is continuous, the extension to all of $\mathcal{U}$ follows.

By Proposition 1.4.2 in Dupuis and Ellis (1997) and Lemma A1, we then obtain the extended multiplier representation. ■

Proof of Observation 1:

The proof of Theorem 1 already showed that the probability measure q is unique and that the utility function u is unique up to positive affine transformations. We also know that for $A > 0$ and $B \in \mathbb{R}$, $\varphi' = A\varphi + B$. Because $e^{-\sigma' u'} = e^{-\sigma'(\alpha u + \beta)} = e^{-\beta} e^{-\alpha\sigma' u}$, it follows from the uniqueness properties of φ that $\sigma' = \frac{1}{\alpha}\sigma$. ■

Proof of Theorem 2:

Ambiguity aversion states that preferences are convex. Hence it is equivalent to a concave representation. Since u is linear with respect to mixture of lotteries, ambiguity aversion is equivalent to the SOEU with φ concave, which means $\sigma \geq 0$. The opposite reasoning applies to ambiguity seeking. ■

Proof of Theorem 3:

(2) ⇒ (1) is trivial. Assume (1). It implies $u_1 \approx u_2$ (Ghirardato and Marinacci, 2002, Proposition 11). We scale utility such that $u_1 = u_2$. Recode lotteries into expected utilities. Using the second-order expected utility formulation of extended variational preferences and the results of Yaari (1969), we immediately obtain $q_1 = q_2$ and φ_2 more concave than φ_1, which implies $\sigma_1 \leq \sigma_2$. ■

Proof that there is a unique solution σ for each value of r.

$f_Y \sim f_P$ and $f_P \sim 15_r 0$ jointly imply $\varphi_\sigma(15r) = ½\varphi_\sigma(15) + ½\varphi_\sigma(0)$, which is equivalent to $15r = ½(15) + ½(0)$ if $\sigma = 0$ and to $\exp(-15\sigma r) = ½\exp(-15\sigma) + ½\exp(0)$ otherwise. Hence,

$$r = ½ \text{ if } \sigma = 0$$

$$r = -\frac{\ln(½\exp(-15\sigma) + ½)}{15\sigma} \text{ if } \sigma \neq 0$$

The proof that r is continuous and decreasing as a function of σ is elementary. By the intermediate value theorem, there is a unique solution σ for each $r \in (0,1)$.

■

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