

Prudence with respect to ambiguity

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Abstract

Under expected utility, prudence is equivalent to a positive third derivative of utility and plays a crucial role in precautionary saving behaviour. Eeckhoudt and Schlesinger (2006) proposed behavioural definitions of prudence and of higher order risk preferences. The present paper proposes a similar definition for prudence with respect to ambiguity, i.e., situations in which objective probabilities are not available. Implications for several ambiguity models are derived. Ambiguity prudence is implied by Hansen and Sargent's (2001) multiplier preferences, empirically correlates with financial behaviour, and plays a key role in prevention behaviour.

Keywords: Ambiguity; prudence; Ellsberg paradox; precautionary saving; prevention

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1 Introduction

The role of higher-order risk preferences (such as risk prudence) in saving and financial behaviour has long been recognised (Leland, 1968; Sandmo, 1970; Kimball; 1990). Risk prudence, which is equivalent to a positive third derivative of utility in expected utility, was initially identified in simple two-period models as necessary and sufficient for precautionary saving, i.e., for future risks to increase saving. A positive third derivative

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of utility is also known to generate a preference for positive skewness (Arditti, 1967), and Kraus and Litzenberger (1976) proposed adapting the CAPM to account for such preference. Risk temperance (Kimball, 1992) and edginess (Lajeri-Chaherli, 2004), which are equivalent to negative and positive fourth and fifth derivatives of u , respectively, have also been discussed in terms of their impact on saving and financial behaviour.

Eeckhoudt and Schlesinger (2006, E&S henceforth) and Eeckhoudt, Schlesinger, and Tsetlin (2009) define these concepts and higher orders of risk preferences, which they call *risk apportionments of order n* , in terms of elementary behavioural properties. Their definitions do not depend on the model in which these concepts are used, exactly as risk aversion (order 2 of risk preferences) can be defined in terms of preferences between lotteries without committing to expected utility or any other decision theory. For instance, risk prudence (order 3) is defined as (always) preferring a 50-50 lottery that provides either a sure loss or a zero-mean lottery to another 50-50 lottery that provides either nothing or both the sure loss and the zero-mean lottery. A prudent decision maker does not want to bear the two "harms" (loss and risk) simultaneously.

Such definitions require a framework in which objective probabilities are available. Knight (1921) noted that in many situations, such probabilities are not available, calling this case unmeasurable uncertainty. Savage (1954) suggested that a decision maker's behaviour can be represented by subjective probabilities together with expected utility. However, Ellsberg's (1961) classic work showed that a lack of knowledge of probabilities (*ambiguity*) has a specific impact on people's behaviour, making Savage's expected utility fail descriptively. Most people prefer situations in which probabilities are known to situations in which they are not. This phenomenon is called ambiguity aversion. In the present paper, a definition of ambiguity prudence will be proposed, and its implications for some of the most prominent ambiguity models will be derived. Ambiguity prudent decision makers will not like bearing two harms simultaneously (exactly like risk prudent decision makers), but these two harms will be a decrease in the probability of a good outcome and additional ambiguity (instead of a loss and additional risk as in the risk prudence definition).

The relevance of ambiguity prudence is threefold. First, it helps better understand and select among ambiguity models. A wide variety of models have been proposed to accommodate ambiguity aversion, but ambiguity aversion alone cannot help to test these

models or differentiate them. Ambiguity prudence can. We will see which specifications of the most famous ambiguity models used in finance and economics imply ambiguity prudence (see section 4).

Second, empirical research has so far mainly highlighted the prevalence of ambiguity aversion through laboratory experiments but only a few studies have demonstrated the external validity of ambiguity aversion outside the lab (Trautmann and van de Kuilen, forthcoming). Interestingly, Dimmock et al. (2015) found that an index related to ambiguity prudence better predicts the financial behaviour of a representative sample of the Dutch population than ambiguity aversion does (see section 6.1). Going beyond the concept of ambiguity aversion is therefore a useful complement to existing empirical studies.

Third, ambiguity prudence can play a crucial role in the application of ambiguity models. Recent results by Guerdjikova and Sciuba (2015) highlighted the key role of ambiguity prudence in the survival of ambiguity-averse agents in a market. In the present paper, we will determine the conditions under which an ambiguity prudent agent facing two threats will decide to exert prevention efforts against the threat she knows less about (section 5.1). We will also introduce the concept of precautionary prevention,¹ defined as the increase of an agent's prevention level due to ambiguity in a risk she is facing. The mere assumption of ambiguity aversion does not say much about the existence of precautionary prevention, but ambiguity prudence is necessary and sufficient for precautionary prevention (section 5.2). Hence, in terms of real-life behaviour, ambiguity prudence means exercising more effort to decrease the probability of a negative scenario when this probability is ambiguous.

The main text of this paper will propose a definition of ambiguity prudence and demonstrate why it matters. The next section introduces the framework and reports E&S's results as a benchmark. Section 3 is dedicated to defining ambiguity prudence. Section 4 establishes which specifications of widely used ambiguity models imply ambiguity prudence, and section 5 illustrates the role of ambiguity prudence in prevention behaviour. A general discussion of the results is presented in section 6, and section 7 concludes the paper. The appendix provides all proofs. An online appendix proposes definitions of any order of ambiguity apportionment in the same way that E&S defined the risk apportionment of order n for all n . An illustration of order 4 (ambiguity temperance) is provided in section 6.4.

¹I am grateful to a referee for suggesting this application.

2 Framework

2.1 Notation

Uncertainty is modelled by a *state space* S , finite or infinite, that contains all possible *states of the world*, typically denoted s . Only one state is true, but the decision maker does not know which one. An *event* is a subset E of S . E^c indicates the complement of E , and the set of all events is a sigma-algebra Σ .

The decision maker under consideration has a deterministic, positive *initial wealth* and will bear the monetary consequences of her decisions. Her decisions may lead to an *outcome* α , any real number such that the *final wealth* (equal to the initial wealth added to the actual outcome) will always be positive. Therefore, the *final wealth domain* is assumed to be \mathbb{R}_{++} .

The decision maker faces situations in which she knows the probability of each possible outcome; we will refer to these situations as *lotteries*, typically denoted x and y . With slight abuse of the notation, we will refer to a degenerate lottery, which yields one unique outcome, as the outcome itself. *Zero-mean lotteries* will be lower indexed with a zero (e.g., x_0 satisfies $E[x_0] = 0$). Consider two lotteries x and y . The lottery $x_p y$ is called a *mixture* of x and y and yields lottery x with probability p and lottery y otherwise. The mixture $x_{\frac{1}{2}} y$ can thus be seen as a fair-coin flip to determine whether the decision maker receives x or y .

The *preferences* \succsim of the decision maker are defined over the set of (*Anscombe-Aumann*) *acts* f , which are finite and Σ -measurable functions assigning lotteries to states such that $f(s)$ is a lottery. Anscombe-Aumann acts have two stages. The first stage can be represented as a tree in which each branch corresponds to a state of the world. At the end of each first-stage branch, the second stage is a lottery, which could also be represented as a tree displaying a probability and ending with an outcome. Some acts assign a unique lottery to all states of the world (i.e., $f(s) = x \forall s \in S$). In such an act, the second stage is always the same (therefore, the first stage does not matter). We will refer to such an act (f satisfying $f(s) = x \forall s \in S$) as the lottery itself (x).

Finally, we will make an assumption on the richness of the state space. In a nutshell, this assumption will guarantee that the decision maker is not limited to extreme beliefs (e.g., one event being judged almost certain). Such extreme beliefs would drive most of

the decision maker's behaviour (she would simply choose the act that assigns the best lottery to the almost certain event) and would make it difficult to draw any conclusions. In the appendix, a weaker condition is used, but it necessitates heavier notation.

Richness Assumption 1. *There exists a partition of S into four events E_1, E_2, E_3, E_4 and a lottery $x \succ 0$ such that $f_1 \sim f_i$ for all $i \in \{2, 3, 4\}$ with f_i defined by $f_i(s) = x$ if $s \in E_i$ and $f_i(s) = 0$ otherwise for all $i \in \{1, \dots, 4\}$.*

2.2 Benchmark: risk preferences

Let us begin with the definitions of E&S. Preferences are *monotone* if $0 \succsim -\kappa$ for all $-\kappa < 0$ and all initial wealth levels, and they are *risk averse* if $0 \succsim x_0$ for all zero-mean lotteries x_0 and all initial wealth levels. Monotonicity and risk aversion are the first and second orders of risk preferences, respectively. Now, consider $(-\kappa)_{\frac{1}{2}}x_0$, which is a mixture that provides a sure loss $-\kappa$ with probability $\frac{1}{2}$ and a zero-mean lottery otherwise. By contrast, $0_{\frac{1}{2}}(x_0 - \kappa)$ provides both the sure loss and the zero-mean lottery simultaneously with probability $\frac{1}{2}$ and nothing otherwise. The decision maker can perceive such a choice situation as beginning with a 50-50 lottery giving $-\kappa$ or 0 and being required to add a zero-mean lottery. She can choose to combine good with bad (adding the extra risk where there is no loss) or good with good (adding randomness where there is already a loss). The decision maker's preferences satisfy *risk prudence* (the third order of risk preferences) if $(-\kappa)_{\frac{1}{2}}x_0 \succsim 0_{\frac{1}{2}}(x_0 - \kappa)$ for all $-\kappa < 0$, all zero-mean lottery x_0 , and all initial wealth levels. As noted by E&S, such a decision maker does not want to bear two "harms" (a loss and a zero-mean risk) simultaneously and prefers $(-\kappa)_{\frac{1}{2}}x_0$, which leads to only one of them, at most.

Risk prudence gives clear results in an *expected utility* framework, in which it is assumed that there exists a utility function u such that preferences over lotteries x are represented by $Eu(\omega + x)$, where ω is the initial wealth. The function u is defined over the final wealth domain, and we will assume, throughout the paper, that it is continuous and strictly increasing.

Theorem (E&S). *Assume expected utility, with u three times differentiable. Then, risk aversion is equivalent to $u'' \leq 0$, and risk prudence is equivalent to $u''' \geq 0$.*

3 Defining ambiguity prudence

Risk arises when the decision maker does not know which outcome she will obtain. Ambiguity arises when the decision maker lacks knowledge of the probability of obtaining each outcome. Hence, one way to study ambiguity consists of applying the definitions developed for risk to probabilities instead of outcomes. This is possible in the Anscombe-Aumann framework, in which states of the world can yield lotteries (and, therefore, probabilities of obtaining an outcome). Let x and y be any lotteries such that $x \succsim y$, p is a probability, and $-\kappa < 0$. If we require monotonicity with respect to probabilities (instead of outcomes), we find that for all such x , y , p , and κ , $x_p y \succsim x_{p-\kappa} y$, which is a form of stochastic dominance. This can be seen as the first order of ambiguity preferences. Throughout, we will refer to the change from $x_p y$ to $x_{p-\kappa} y$ as a *probability loss*. Let us now turn to the second order, ambiguity aversion.

3.1 Ambiguity aversion

We start from a simplified version of Ellsberg's (1961) paradox. Consider two urns. The first urn ("known") has one red and one black ball. The second urn ("unknown") also has two balls, but we do not know whether they are both red or both black.² The decision maker can win \$10 if she draws a red ball, and she must choose from which urn to draw. Most people tend to prefer the known urn to the unknown urn. Then, the decision maker is asked from which urn she would like to draw, if she can now win \$10 if she picks a black ball. Most people would again prefer to draw from the known urn. Under expected utility, the first choice reveals that the probability of drawing a red ball is lower in the unknown urn than in the known urn, but the second choice reveals that the probability of drawing a black ball is also lower in the unknown urn than in the known urn. This leads to a contradiction because the sum of the probabilities of drawing a red ball and of drawing a black ball should be 1 for both urns. The preference for the known urn is attributed to *ambiguity aversion*, the dislike of not knowing the probability of winning when drawing from the unknown urn.

In this example, the ambiguity comes from the composition of the unknown urn (the

²In the original Ellsberg paradox, the known urn has 50 red and 50 black balls, and the unknown urn can be any 100-ball urn consisting of red and black balls. This is further discussed in section 6.4 and in the online appendix.

known urn being risky, with known probabilities). If we denote as R the event "the two balls in the unknown urn are red", we can represent the simplified Ellsberg paradox in tables in which rows display the lottery assigned by the acts to the events indicated at the top of each column.

	R	R^c		R	R^c
Win if the ball is red	$10_{\frac{1}{2}}0$	$10_{\frac{1}{2}}0$	\succsim	10_10	10_00
Win if the ball is black	$10_{\frac{1}{2}}0$	$10_{\frac{1}{2}}0$	\succsim	10_00	10_10
	Drawing from the known urn			Drawing from the unknown urn	

Table 1: Simplified Ellsberg paradox

The first row refers to the case in which red is the winning colour, the second row to the case in which black is the winning colour. The left-hand table corresponds to drawing a ball from the known urn, which gives a 50% chance of winning no matter what the unknown urn contains. In the right-hand table, corresponding to drawing a ball from the unknown urn, the probability of winning is 100% or 0%, depending on the composition of the unknown urn and on the colour one must draw. If we observe that the unknown urn is preferred for the first choice, it may still be that the decision maker under consideration is ambiguity averse but believes that the unknown urn is more likely to contain red balls than black balls. However, someone who would strictly prefer the right-hand acts for both rows is surely not ambiguity averse. We should therefore require that a decision maker exhibit at least one of the two preferences to be called ambiguity averse.

The Ellsberg paradox entails preferences over permuted bets. The second row indeed permutes the lotteries over the events. This permutation makes it possible to control for beliefs. If choosing the left-hand act in the first choice is only due to beliefs about the column events, permuting the lotteries should change the preference. If it does not, not only does it reject the belief explanation, it also rejects ambiguity-seeking behaviour. The following definition generalises this intuition.

Definition 1. *A decision maker is ambiguity averse if for all initial wealth levels, events E and E^c , lotteries $x \succsim y$, and probabilities p and ε , at least one of the following two preferences hold:*

E	E^c	\succsim	E	E^c
$x_p y$	$x_p y$		$x_{p+\varepsilon} y$	$x_{p-\varepsilon} y$
$x_p y$	$x_p y$	\succsim	$x_{p-\varepsilon} y$	$x_{p+\varepsilon} y$
$x_p y$	$x_p y$		$x_{p-\varepsilon} y$	$x_{p+\varepsilon} y$

Table 2: Ambiguity aversion

Definition 2. A decision maker exhibits ambiguity neutrality of order 2 if for all initial wealth levels, events E and E^c , lotteries $x \succsim y$, and probabilities p and ε , one of the two preferences of Table 2 holds and the other is reversed.

Definition 3. A decision maker exhibits strict ambiguity aversion if she exhibits ambiguity aversion but not ambiguity neutrality of order 2.

If, in the simplified Ellsberg paradox presented above, the known urn is strictly preferred in both choices, then we can reject ambiguity neutrality of order 2, and we have evidence for strict ambiguity aversion.

Several alternative definitions of ambiguity aversion exist. The most famous definition is Schmeidler's (1989) (also used by Gilboa and Schmeidler, 1989, Maccheroni et al., 2006, Strzalecki, 2011, and many others). It states that preferences are convex with respect to mixtures, i.e., if two acts f and g are indifferent, then an act that gives, for each state of the world s , the lottery $f(s)_p g(s)$, is preferred to the acts f and g themselves. Chateauneuf and Tallon (2002) proposed a definition in which f and g (and possibly other acts that are indifferent to f and g) are complete hedges. Acts are complete hedges when a linear combination exists that gives the same lottery for all $s \in S$. In Chateauneuf and Tallon's (2002) definition of ambiguity aversion (or *preference for diversification*), the completely hedged act (i.e., the lottery) is preferred to any of the acts it is based on.

Siniscalchi (2009) considered the case in which f and g can be completely hedged by a $\frac{1}{2}$ mixture. He called f and g *complementary acts* if there exists x such that $f(s)_{\frac{1}{2}} g(s) = x$ for all $s \in S$. *Simple diversification* holds if $f \sim g$ with $f(s)_{\frac{1}{2}} g(s) = x$ implies $x \succsim f$. *Complementary ambiguity aversion* states that if f and g are complementary acts with x as their hedge, $f \sim y$, and $g \sim z$, then $x \succsim y_{\frac{1}{2}} z$. Definition 1 is very close to complementary ambiguity aversion. Let us call f and g the right-hand acts of Table 2. These acts are complementary, with their $\frac{1}{2}$ mixture x being the left-hand acts. Definition 1 requires that such complementary acts cannot be both preferred to their $\frac{1}{2}$ mixture.

Klibanoff et al. (2005) proposed a definition of ambiguity aversion as aversion to mean-

preserving spread in terms of utility units. The intuition of the definition proposed above is also close to that of Klibanoff et al. (2005). Ghirardato and Marinacci (2002) proposed defining ambiguity aversion as being more ambiguity averse than an expected utility maximiser. In practice, this implies that an ambiguity-averse agent cannot prefer act f to lottery x if an expected utility agent with the same utility prefers x to f . Epstein (1999) does not use expected utility but uses probabilistic sophistication as a benchmark. In contrast with these definitions, definition 1 is directly and easily implementable in an experimental setup or a survey (without having to reconstruct the preferences of a benchmark agent).

3.2 Ambiguity prudence

We now turn to order-3 ambiguity preferences. Risk prudence was defined as a preference for assigning a loss to the part of an act in which there was no risk instead of bearing the two "harms" simultaneously. Applying the same logic to possible probabilities instead of possible outcomes, ambiguity prudence can be defined as the preference for assigning a probability loss to a part of an act in which there is no ambiguity about the probability of winning. In short, the decision maker prefers improving her odds in ambiguous events rather than in unambiguous events.

Definition 4. *A decision maker is ambiguity prudent if for all initial wealth levels, events $E_{11}, E_{12}, E_{21}, E_{22}$ partitioning S , lotteries $x \succsim y$, and probabilities p, ε , and κ , at least one of the following four preferences hold:*

E_{11}	E_{12}	E_{21}	E_{22}		E_{11}	E_{12}	E_{21}	E_{22}
$x_{p-\kappa}y$	$x_{p-\kappa}y$	$x_{p+\varepsilon}y$	$x_{p-\varepsilon}y$	\succsim	$x_p y$	$x_p y$	$x_{p-\kappa+\varepsilon}y$	$x_{p-\kappa-\varepsilon}y$
$x_{p-\kappa}y$	$x_{p-\kappa}y$	$x_{p-\varepsilon}y$	$x_{p+\varepsilon}y$	\succsim	$x_p y$	$x_p y$	$x_{p-\kappa-\varepsilon}y$	$x_{p-\kappa+\varepsilon}y$
$x_{p+\varepsilon}y$	$x_{p-\varepsilon}y$	$x_{p-\kappa}y$	$x_{p-\kappa}y$	\succsim	$x_{p-\kappa+\varepsilon}y$	$x_{p-\kappa-\varepsilon}y$	$x_p y$	$x_p y$
$x_{p-\varepsilon}y$	$x_{p+\varepsilon}y$	$x_{p-\kappa}y$	$x_{p-\kappa}y$	\succsim	$x_{p-\kappa-\varepsilon}y$	$x_{p-\kappa+\varepsilon}y$	$x_p y$	$x_p y$

Table 3: Ambiguity prudence

Let us observe the probabilities used to obtain mixtures of x and y and consider the lottery $x_p y$ as the benchmark. In two columns of the left-hand table, the probability of getting the preferred lottery x is $p - \kappa$, which indicates a probability loss. In the other two columns, the probability of getting x is ambiguous: it can be $p - \varepsilon$ or $p + \varepsilon$ (as in

the definition of ambiguity aversion). In the first two columns on the right-hand table, the probability of getting x remains equal to the benchmark (p), but in the other two columns, the probability loss ($-\kappa$) is combined with the ambiguity ($\pm\varepsilon$). The dislike of bearing both harms simultaneously should favour the left-hand table.

However, it could still be that, for a given row, the decision maker prefers the right-hand table. For instance, if she thinks that E_{21} and E_{22} are impossible, then she would prefer the right-hand act in the first row. Permuting the lotteries between E_{11} and E_{12} and between E_{21} and E_{22} (thus obtaining the second row) would not change her preference for the right-hand table. But if she continues preferring the right-hand table for the third and the fourth row, then her beliefs alone could not explain this behaviour and we should not call her ambiguity prudent. This is why the definition requires that at least one of the four preferences is in favour of the left-hand table. We now define neutrality and strict ambiguity prudence.

Definition 5. *A decision maker exhibits ambiguity neutrality of order 3 if for all initial wealth levels, events E and E^c , lotteries $x \succsim y$, and probabilities p , ε , and κ , at least one of the preferences of Table 3 holds and at least one other is reversed.*

Definition 6. *A decision maker exhibits strict ambiguity prudence if she exhibits ambiguity prudence but not ambiguity neutrality of order 3.*

The following example, based on Figure 1, illustrates these definitions. Imagine that one card is randomly drawn from a deck of four cards. These cards are either all red or all blacks. One card is a King, one card is a Queen, and the two others are either both Jacks or both Aces. This deck thus entails two sources of ambiguity: the colour of the cards (red or black) and the type of two of the four cards (Jacks or Aces). The decision maker can bet on combinations of types and colours. She wins \$10 if the card drawn from the bag is a red Queen, a black Queen, a red King or a black Ace. She is told that she must give up one winning possibility (probability loss): red Queen or black Queen. Choosing the colour of the Queen to abandon means choosing whether to allocate the probability loss to a case in which the probability of winning is known (red, because she knows there is a King) or to a case in which it is ambiguous (black, because the number of Aces is unknown).

To summarise, she has two options:

- Option A: \$10 if the card is a red King, a black Ace, or a *red Queen*.
- Option B: \$10 if the card is a red King, a black Ace, or a *black Queen*.



Figure 1: A deck of four cards

(The cards are either all red or all black. There is one King and one Queen, and the two other cards are either two Aces or two Jacks.)

The states of the world, which determine the probability of winning, are the possible contents of the deck. Let us denote these four states RA , RJ , BA , and BJ , where R indicates that the cards are red, B indicates that they are black, A indicates that two cards are Aces, and J indicates that two cards are Jacks. If the cards are red and two are Aces (event RA), option A gives a 50% chance to win, and option B gives a 25% chance to win. Table 4 indicates the lottery that each option yields for each event (each possible composition of the deck). It corresponds to the first row of the tables of definition 7, with $x = 10$, $y = 0$, $\kappa = \varepsilon = 0.25$, and $p = 0.5$.

RA	RJ	BA	BJ	\succsim	RA	RJ	BA	BJ
$10_{0.25}0$	$10_{0.25}0$	$10_{0.75}0$	$10_{0.25}0$		$10_{0.5}0$	$10_{0.5}0$	$10_{0.5}0$	10_0
Option B					Option A			

Table 4: Example of ambiguity prudence

For option A, the first two columns show a 50% chance of winning, but the extreme right columns indicate both a probability loss (-25% , i.e., giving up black-Queen winning possibility) and some ambiguity ($\pm 25\%$, because of the unknown number of Aces). The first two columns of option B correspond to a probability loss (giving up red-Queen winning possibility), and the other two show some ambiguity ($\pm 25\%$). Ambiguity prudence is defined as the dislike of bearing the two "harms" (probability loss and ambiguity) at the same time and therefore preferring option B (unless one has exotic beliefs about the deck composition). To obtain the other three rows of the tables of definition 4, it suffices to replace red with black and/or Aces with Jacks in options A and B. If option B is always strictly preferred, ambiguity neutrality of order 3 is rejected, and evidence is provided for strict ambiguity prudence.

4 Ambiguity models and prudence

4.1 Two-stage expected-utility models

The most famous model in the category of two-stage expected-utility models was proposed by Klibanoff et al. (2005). For simplicity, we will begin with another, simpler one. We define *second-order expected utility* as introduced by Neilson (2010): there exists a utility function u defined on the final wealth domain, another function φ defined on the image of u , and a subjective probability distribution P such that preferences over acts f are represented by $\int_S \varphi(Eu(\omega + f(s))) dP(s)$. The function φ captures deviations from expected utility caused by ambiguity. If φ is linear, it is equivalent to (Anscombe-Aumann (1963)) expected utility $\int_S Eu(\omega + f(s))dP(s)$.

Theorem 1. *Assume Richness Assumption 1 and second-order expected utility, with φ strictly increasing and three-times differentiable. Then ambiguity aversion is equivalent to $\varphi'' \leq 0$, and ambiguity prudence is equivalent to $\varphi''' \geq 0$.*

The proof of this theorem is provided in the appendix. To show the necessity of ambiguity aversion for $\varphi'' \leq 0$, we can first remark that, in the definition, a lottery assigned to event E in the first row is assigned to event E^c in the second row and conversely. Hence, scaling u such that $Eu(\omega + x) = 1$ and $Eu(\omega + y) = 0$, the sum of the value of the two left-hand acts is $2\varphi(p)$, and the sum of the value of the two right-hand acts is $\varphi(p-\varepsilon) + \varphi(p+\varepsilon)$. The subjective probability P dropped out. This shows how considering two preferences enables us to control for beliefs. Then, E&S's theorem can be applied to φ , with p playing the role of initial wealth and $\pm\varepsilon$ the role of the zero-mean lottery. If both right-hand acts were preferred to their left counterparts, φ would not be concave. The richness assumption is used to obtain the sufficiency result. The proof of the equivalence between ambiguity prudence and $\varphi''' \geq 0$ follows the same steps.

Corollary 1. *Expected utility (second-order expected utility with φ linear) implies the ambiguity neutrality of orders 2 and 3.*

It is well known that (Anscombe-Aumanns) expected utility does not allow for ambiguity aversion. This corollary confirms that it does not allow for ambiguity prudence either.

For simplicity, Neilson's (2010) second-order expected utility was presented first, but similar results apply to Klibanoff et al.'s (2005) smooth model of ambiguity attitude. Let us denote $\Delta(S)$ the set of the probability measures for the state space S . Under the *smooth model*, there exists a utility function u defined for monetary outcomes, another function φ defined on the image of u , and a subjective (second-order) probability distribution μ for $\Delta(S)$ such that preferences over acts f are represented by $\int_{\Delta(S)} \varphi \left(\int_S Eu(\omega + f(s))dP(s) \right) d\mu(P)$. A complication of the smooth model is that the second-order probability distribution may create interdependencies between the possible expected utilities $\int_S Eu(\omega + f(s))dP(s)$. In E&S's definition of risk prudence, the 50-50 mixture and the zero-mean lottery x_0 are supposed to be independent. In Neilson's model, the separability of the model over the four events and the permutations used in definition 4 ruled out the interdependencies of expected utilities. This does not work anymore for the smooth model. Hence, in the following theorem, we specify a state space that is the Cartesian product of two state spaces and assume that the decision maker's support for his second-order beliefs μ is $\Delta(S) = \Delta(S^1) \times \Delta(S^2)$. In the example above, S^1 describes the uncertainty related to the colour of the cards and S^2 is the uncertainty about the presence of Aces or Jacks. We first adapt the richness assumption needed to derive the results and then state the theorem for the smooth model.³

Richness Assumption 2. *Let $S = S^1 \times S^2$. For both $i \in \{1, 2\}$, there exist two complementary events $E_1^i, E_2^i \subset S^i$ such that for all lotteries x, y, z, t , $f_1^1 \sim f_j^i$ for all $i, j \in \{1, 2\}$ with f_j^i assigning x to E_j^i , y to E_{3-j}^i , z to E_j^{3-i} and t otherwise.*

Theorem 2. *Let $S = S^1 \times S^2$ and $\Delta(S) = \Delta(S^1) \times \Delta(S^2)$, and let any events of Definition 4 be such that $E_{ij} = E_i^1 \times E_j^2$ for some $E_i^t \subset S^t$. Assume Richness Assumption 2 and the smooth model with φ strictly increasing and three-times differentiable. Then, ambiguity aversion is equivalent to $\varphi'' \leq 0$, and ambiguity prudence is equivalent to $\varphi''' \geq 0$.*

4.2 Multiple-prior models

In the two-stage expected utility model, the decision maker holds beliefs over possible probability distributions. In multiple-prior models, the decision maker considers a subset C of the set of all possible probability measures $\Delta(S)$ and then maximises the worst

³Huang et al. (2015) propose using the index $\frac{\varphi'''}{\varphi'}$ to capture the intensity of ambiguity prudence and to compare decision makers.

expected utility she might obtain (*maxmin expected utility*, Gilboa and Schmeidler, 1989) or a linear combination of the worst and the best expected utility (α -*maxmin*, Ghirardato et al., 2004). Formally, under α -maxmin, an act f is represented by

$$\alpha \times \min_{P \in C} \left(\int_S Eu(\omega + f(s)) dP(s) \right) + (1 - \alpha) \times \max_{P \in C} \left(\int_S Eu(\omega + f(s)) dP(s) \right).$$

Maxmin expected utility corresponds to $\alpha = 1$. The set of priors C is subjective and gives flexibility to the multiple-prior models. In financial applications, a typical form of multiple priors is called ϵ -contamination (Epstein and Wang, 1994). The decision maker has a probability measure Q in mind but considers possible perturbations of Q . The degree of perturbations (or errors) is ϵ , which satisfies $0 \leq \epsilon \leq 1$. If the set of perturbing probabilities is $\Delta(S)$ itself, $C = \{P : P = (1 - \epsilon)Q + \epsilon T, T \in \Delta(S)\}$. Under this specification, the theorem below establishes that ambiguity prudence must hold no matter what α is. By contrast, ambiguity aversion does depend on α . It is equivalent to assigning more weight to the worst expected utility than to the best expected utility ($\alpha \geq 1/2$).

Theorem 3. *If preferences are represented by α -maxmin with*

$C = \{P : P = (1 - \epsilon)Q + \epsilon T, T \in \Delta(S)\}$ for a given probability measure Q and a contamination $0 \leq \epsilon \leq 1$, then ambiguity aversion is equivalent to $\alpha \geq 1/2$ and ambiguity prudence holds. Furthermore, if $\epsilon > 0$, strict ambiguity prudence holds.

4.3 Multiplier preferences

Multiplier preferences were introduced by Hansen and Sargent (2001) to model the behaviour of a decision maker (for instance, a policy maker) who has a baseline probability measure in mind but wants to implement a decision that is robust to the possibility that the real probability differs from the baseline probability. However, probability measures deviating from the baseline are judged to be less realistic and are therefore penalised by how far they are from the baseline (relative entropy captures this distance). Formally, there exists a utility function u , a set $\Delta(S)$ of probability measures for S , and a subjective probability distribution $Q \in \Delta(S)$ such that preferences are represented by $\min_{P \in \Delta(S)} \left(\int_S Eu(\omega + f(s)) dP(s) + \theta R(P||Q) \right)$, where $\theta \in (0, \infty]$ is an ambiguity aversion coefficient and $R(P||Q)$ is the relative entropy of P with respect to Q .

A more general class of preferences, called variational preferences, was axiomatised by Maccheroni et al. (2006), and multiplier preferences were axiomatised by Strzalecki (2011),

who showed that multiplier preferences coincide with the intersection of variational preferences and preferences that can be represented by second-order expected utility. Strzalecki (2011) derived his result from the fact that $\min_{P \in \Delta(S)} (\int_S Eu(\omega + f(s))dP(s) + \theta R(P||Q))$ is ordinally equivalent to $\int_S \varphi_\theta (Eu(\omega + f(s))) dQ(s)$ with

$$\varphi_\theta(u) = \begin{cases} -\exp(-\frac{u}{\theta}) & \text{for } \theta < \infty, \\ u & \text{for } \theta = \infty. \end{cases} \quad (1)$$

One can immediately see that $\varphi''_\theta \leq 0$ and $\varphi'''_\theta \geq 0$. Then, we have the following:

Theorem 4. *If preferences are multiplier preferences, then they exhibit ambiguity aversion and ambiguity prudence. If Richness Assumption 1 holds and $\theta < \infty$, then strict ambiguity aversion and strict ambiguity prudence hold.*

4.4 Choquet expected utility

Under Schmeidler's (1989) *Choquet expected utility*, there exists a utility function u defined for monetary outcomes and a non-additive weighting function W defined on Σ (satisfying $W(\emptyset) = 0$, $W(S) = 1$, and $W(A) \leq W(B)$ if $A \subset B$) such that preferences over acts f are represented by $\int_S Eu(\omega + f(s))dW(s)$. We say that W is *binary superadditive* if for all E , $W(E) + W(E^c) \leq 1$. Binary superadditivity suggests that the decision weights of complementary events are "too low", capturing some form of pessimism. Following Tversky and Wakker (1995), we say that W is *lower subadditive* if $W(E) + W(F) \geq W(E \cup F)$ for all E and F such that $W(E \cup F)$ is bounded away from 1; we say that W is *upper subadditive* if $1 - W(S - E) \geq W(E \cup F) - W(F)$ for all E and F such that $W(F)$ is bounded away from 0. Lower and upper subadditivity captures the fact that event E has a greater impact when added to the null event or subtracted from the universal event than when added to F or subtracted from $E \cup F$. Intuitively, people are more sensitive to departures from impossibility or certainty than to changes concerning events that are neither impossible nor certain.

Theorem 5. *If preferences are represented by Choquet expected utility, then a binary superadditive W implies ambiguity aversion. If, further, there exists $\eta > 0$ such that $\eta < W(E) < 1 - \eta \forall E \in \Sigma \setminus \{\emptyset, S\}$, a lower and upper subadditive W implies ambiguity prudence.*

Theorem 5 relates typical properties of weighting functions to ambiguity aversion and prudence. In this theorem, the existence of η guarantees that the weight assigned by W to any (non-empty and non-universal) event is bounded away from 0 and 1 such that the definition of upper and lower subadditivity can be used without restrictions.

A special case of Choquet expected utility, called *neo-additive*, was proposed by Chateauneuf et al. (2007), in which W is an affine transformation of a probability measure P over S , such that $W(E) = \frac{a-b}{2} + (1-a) \times P(E)$ for all events E satisfying $0 < P(E) < 1$ and with $a \in [0, 1]$ and $b \in [-a, a]$.

A decision maker with $a > 0$ will react less to a change of probabilities within the range $(0, 1)$ than an expected utility maximiser ($a = 0$). Parameter a captures the limited ability of the decision maker to discriminate between likelihood levels (i.e., her insensitivity to likelihood) (Figure 2a). This phenomenon has been called *ambiguity-generated insensitivity (a-insensitivity)* by Maafi (2011) and Baillon et al. (2012) and is discussed by Wakker (2010, p.332). The second parameter, b , captures a form of pessimism, typically interpreted as ambiguity aversion. The higher b is, the less weight will be given to the best lottery (in *Eu*-terms) and, therefore, the more weight will be given to the worst lottery (Figure 2b). Abdellaoui et al. (2011) used a and b to study how ambiguity attitudes are influenced by sources of uncertainty.

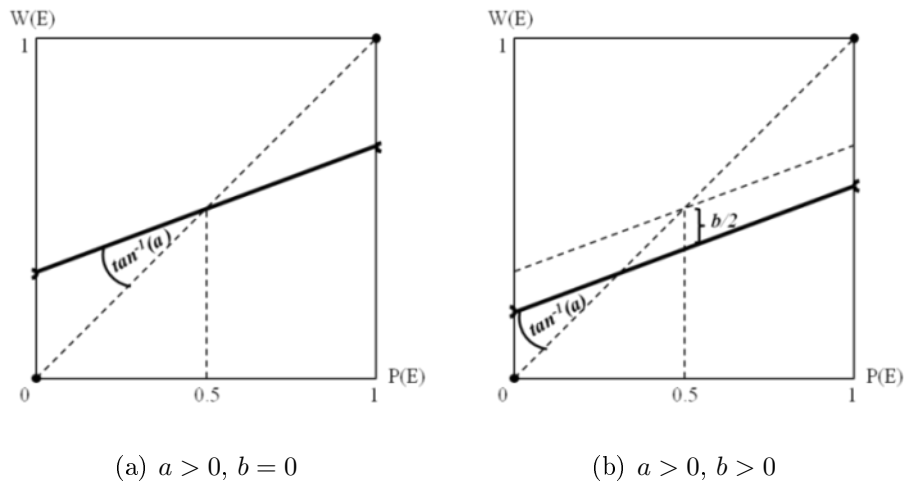


Figure 2: Neo-additive weighting functions
(\tan^{-1} is the inverse of the tangent function.)

The neo-additive Choquet expected utility value of an act f can also be written as

$$(1 - a) \int_S Eu(\omega + f(s))dP(s) + \frac{a-b}{2} \max\{Eu(f(s)) : P(s) > 0\} + \frac{a+b}{2} \min\{Eu(f(s)) : P(s) > 0\} \quad (2)$$

This formula shows that a captures how much weight is (not) given to the expected utility term and that b captures pessimism (the higher b , the more the minimum impacts the decision).

Theorem 6. *If preferences are represented by Choquet expected utility with W neo-additive and Richness Assumption 1 holds, then ambiguity aversion is equivalent to $b \geq 0$. If preferences are represented by Choquet expected utility with W neo-additive and $P(E) > 0 \forall E \in \Sigma \setminus \emptyset$, then ambiguity prudence holds. If, further, Richness Assumption 1 holds and $a > 0$, then strict ambiguity prudence holds.*

Theorem 6 confirms that b captures ambiguity aversion, as used in empirical work. It is worth noting that Schmeidler's (1989) ambiguity aversion (which can be seen as a preference for smoothing Eu values across state spaces) is equivalent to W being convex and, therefore, to $b \geq 0$ and $b = a$ (assuming that the state space is rich enough). The definition of ambiguity aversion proposed in this paper does not restrict a , unlike Schmeidler's definition. Theorem 6 also shows the equivalence between $a > 0$ and strict ambiguity prudence. The restrictions on beliefs ($P(E) > 0 \forall E \in \Sigma \setminus \emptyset$) are necessary to easily identify the worst and best expected utility values the decision makers can obtain. If $a > 0$, then the decision maker cares about the best and the worst expected utility she can obtain and gives them additional weight. From Table 3 in definition 4, we can see that both the best and the worst lotteries in the right-hand table are worse than the best and the worst lotteries of the right hand side table (e.g., if $\varepsilon > \kappa$, the best and worst lotteries are $x_{p+\varepsilon-\kappa}$ and $x_{p-\varepsilon-\kappa}$ in the left-hand table and $x_{p+\varepsilon}$ and $x_{p-\varepsilon}$ in the right-hand table). The result relating a to prudence follows from this.

5 Prevention behaviour

Prevention was initially studied by Ehrlich and Becker (1972). It involves choosing the optimal (costly) reduction of the probability of a loss. An ambiguity prudent agent

will especially dislike situations in which this probability is ambiguous. She will therefore adapt her prevention behaviour.

5.1 Choice of prevention target

In April 2009, a new strain of flu was discovered (informally called swine flu or Mexican flu) and appeared to lead more often to serious complications than the regular (seasonal) flu. The information initially available was very limited, creating much uncertainty. Quickly, governments and health agencies had to decide whether to develop and order vaccines against this new flu, possibly at the cost of not providing vaccines for the seasonal flu. A prevention tool such as a vaccine increases the probability of not getting the flu, but should this "probability gain" be assigned to preventing the new, ambiguous flu or the regular, risky one?

The following stylised example aims to highlight the role of ambiguity prudence in the choice of prevention methods. We consider an agent possibly facing two threats $\{R, N\}$ (e.g., regular flu vs. new flu), which may harm her health. Which threat she will face is unknown, or ambiguous. The agent's utility is normalised to 1 if her health is good and to 0 if her health is bad. One of the threats (R) is well known and leads to a probability p of good health. The other threat (N) is unknown and has two possible scenarios $\{M, L\}$. It may be more dangerous with a probability of good health equal to $p - \eta$ only or actually less dangerous with a probability $p + \eta$ of good health (with $0 < \eta < p$ and $\eta < 1 - p$). A prevention method (e.g., vaccine) can increase, for one of the two threats only, the probability of being in good health by κ (with $0 < \kappa < 1 - p - \eta$).

Table 5 describes the probability of good health for each possible event of the state space $S = \{R, N\} \times \{M, L\}$, depending on the target of the prevention method.

<i>RM</i>	<i>RL</i>	<i>NM</i>	<i>NL</i>	\approx	<i>RM</i>	<i>RL</i>	<i>NM</i>	<i>NL</i>
<i>p</i>	<i>p</i>	<i>p - η + κ</i>	<i>p + η + κ</i>		<i>p + κ</i>	<i>p + κ</i>	<i>p - η</i>	<i>p + η</i>
Prevention against the new threat					Prevention against the regular threat			

Table 5: Choice of prevention

The table shows how this example is related to the definition of ambiguity prudence. The agent may prefer the prevention against the regular threat if she thinks it is much more likely to occur than the new one, and vice versa. Yet, even if she thinks both

threats are equally likely to occur, she will prefer prevention against the new one if she is ambiguity prudent. Specifically, she will prefer prevention against the new threat if she has

- preferences represented by the smooth ambiguity model with $\varphi''' > 0$ and symmetric second-order beliefs ($\mu(P) = \mu(1 - P)$ for all $P \in \Delta(\{R, N\}) \times \Delta(\{M, L\})$);
- α -maxmin preferences with an ϵ -contamination set of priors around Q assigning $\frac{1}{4}$ to the four events of S ;
- multiplier preferences with a baseline probability Q as above;
- preferences represented by neo-additive Choquet expected utility with Q as above.

This example shows the role of ambiguity prudence in the choice of prevention target. The agent prefers to improve her odds in ambiguous events, rather than in unambiguous events. The following sub-section will present a simple two-period model in which an agent will increase her prevention efforts when facing ambiguity if and only if she is ambiguity prudent.

5.2 Precautionary prevention

In addition to vaccines, there are many ways to try to have a healthy future: exercising, eating healthier food, regular health check-ups, etc. Choosing the optimal level of prevention efforts involves a trade-off between the current (monetary) costs and the future benefits, which can be measured as an increase in the probability of being healthy. If an agent perceives the future as more ambiguous, how should her prevention efforts change? For instance, imagine that the agent learns of the discovery of a new technology such as nanotechnology, which may be used in the future for new medical applications but that some claim will create potential health hazards. Should her prevention efforts stay the same?

We study an agent living two periods and whose initial income in both periods is 1. She may lose her income in period 2 (for instance, if she stops working because she becomes ill). In period 1, she can provide (costly) prevention efforts to decrease the probability of the loss in period 2 (see Menegatti, 2009, for such a model). We will study her optimal prevention level (also called self-protection level). Her utility function u for both periods

is normalised such that $u(0) = 0$ and $u(1) = 1$. We denote e as her level of prevention effort and $p(e)$ as the resulting probability of the loss. Compared with the certainty of receiving an income, $p(e)$ can be interpreted as a probability loss as defined earlier, i.e., a decrease in the probability of a good outcome. If we further introduce ambiguity, the model becomes close to the definition of ambiguity prudence, combining a probability loss with ambiguity.

In the no-ambiguity case, the agent maximises:

$$u(1 - e) + (1 - p(e)) \quad (3)$$

We assume that $p' \leq 0$ (an additional effort decreases the probability of the loss), $p'' \geq 0$, $u' \geq 0$, and $u'' \leq 0$. To exclude corner solutions (also when we introduce ambiguity), we assume that there exists η such that $\eta < p(e) < 1 - \eta$ for all e and $-p'(0) > u'(1)$. Her optimal level of prevention activity will thus satisfy $-p'(e^*) = u'(1 - e^*)$. The marginal gain in expected utility in period 2 should equate to the marginal utility loss in period 1.

Imagine that the individual is ambiguity averse and maximises:

$$\varphi(u(1 - e)) + \varphi(1 - p(e)) \quad (4)$$

In such a model, ambiguity aversion ($\varphi'' < 0$) implies a preference for smoothing utility across periods. It will increase the prevention level with respect to the ambiguity neutral case described above if $u(1 - e^*) \geq 1 - p(e^*)$ (i.e., if the first-period utility is higher than the second-period expected utility) and decrease it otherwise. The new optimal effort e^{**} is given by the constraint

$$-p'(e^{**})\varphi'(1 - p(e^{**})) = u'(1 - e^{**}) \times \varphi'(u(1 - e^{**})). \quad (5)$$

Using the concavity of φ and the convexity of p , we can infer that $e^{**} \geq e^*$ is equivalent to $u(1 - e^*) \geq 1 - p(e^*)$.

Assume now the presence of ambiguity in period 2 (e.g., the agent learns the discovery of nanotechnologies). For a given effort e , the probability of the loss in period 2 can be anywhere in $[p(e) - \eta, p(e) + \eta]$ (e.g., whether the medical applications of nanotechnologies happen to outweigh the risks they cause in other domains). The agent does not have any

information about which probability is more likely to occur and has uniform second-order beliefs over $[p(e) - \eta, p(e) + \eta]$. The model with ambiguity aversion becomes:

$$\varphi(u(1 - e)) + \int_{q \in [-\eta, \eta]} \frac{1}{2\eta} \varphi(1 - p(e) + q) dq \quad (6)$$

Let us call *precautionary prevention* an increase in prevention due to ambiguity. If she were ambiguity neutral, she would still maximise equation 3 and the optimal effort would remain e^* . There would be no precautionary prevention. In the ambiguity-averse case, the first order condition becomes:

$$-p'(e^{***}) \int_{q \in [-\eta, \eta]} \frac{1}{2\eta} \varphi'(1 - p(e) + q) dq = u'(1 - e^{***}) \times \varphi'(u(1 - e^{***})). \quad (7)$$

If φ is quadratic (ambiguity neutrality of order 3), the first order condition with ambiguity (equation 7) will be equivalent to equation 5 and the optimal effort level would not change. Second-order preferences (ambiguity aversion) do not predict the impact of future ambiguity on prevention but third-order preferences (ambiguity prudence) do. Indeed, the marginal effect of efforts on the second-period utility (left member of equation 7) is higher than without ambiguity (left member of equation 5) for any η if and only if φ' is convex. If the marginal effect of efforts is higher, the optimal effort level must be higher in equation 7 than in equation 5 to restore equality. Hence, precautionary prevention ($e^{***} \geq e^*$) is equivalent to ambiguity prudence. Even though the ambiguity is symmetric (a new technology bringing both possible good news and bad news), an ambiguity prudent agent assigns more weight to the bad news (i.e., to q negative, implying lower expected utility) than to the symmetric good news through the convexity of φ' .⁴

6 Discussion

6.1 Prudence in empirical work

The definition of ambiguity prudence proposed in section 3 (and the corresponding example) is directly testable in experiments. To date, we can only find indirect evidence of

⁴Prevention is unaffected by ambiguity if the agent uses Choquet expected utility with neo-additive capacities. If there is no ambiguity, the model is equivalent to the ambiguity neutral case (equation 3). With ambiguity, it becomes $u(1 - e) + (1 - p(e) - b\eta)$, but the optimal prevention is not affected by ambiguity.

ambiguity prudence. Theorem 6 provided a formal link between the way the two parameters (a and b) are used in several empirical studies and definitions of ambiguity aversion and prudence. It is especially interesting that Dimmock et al. (2015), who elicited these parameters for a representative sample of the Dutch population, found that parameter a , rather than b , correlated with financial behaviour. In their study, parameter a negatively correlated with stock market participation and private business ownership; that is, the more ambiguity prudent individuals were, the less likely they were to invest in the stock market or to have their own business. It is noteworthy that in a similar sample, Noussair et al. (2012) found that risk prudence also correlated with wealth and saving behaviour. This suggests that studying both risk and ambiguity prudence can be fruitful to better understand saving and financial behaviour since both forms of prudence are related to precautionary saving, for instance. Section 5.2 also predicts an effect of ambiguity prudence on prevention behaviour, offering a possibility to test the external validity of ambiguity prudence.

For long, the literature on risk prudence and higher order risk attitudes remained theoretical. The experimental research is recent and has been facilitated by the definitions proposed by E&S. Risk prudence appeared to be a prevalent phenomenon (Tarazona-Gomez, 2004; Ebert and Wiesen, 2011; Deck and Schlesinger, 2010; Noussair et al., 2012; Maier and Ruger, 2012). It has also been related to preference for skewness by Ebert and Wiesen (2011) and Ebert (2012). Dillenberger and Segal (2013) studied the impact of skewed noise on compound lotteries when decision makers fail to reduce them. The non-reduction of compound lotteries is often associated with ambiguity attitude (Halevy, 2007). Crainich et al. (2013) called mixed risk lovers people who always like to combine "good" with "good" in E&S's lotteries and showed that for such people, all successive derivatives of the utility are positive; therefore, they are also risk prudent. This may explain why risk prudence is a prominent phenomenon. Deck and Schlesinger (2014) provided evidence that risk lovers are also prudent. Their results are compatible with two types of decision makers: those who will always attempt to "combine good with bad" (and will be risk averse, risk prudent, risk temperate) and those who will always attempt to combine good with good (and will therefore be risk loving, risk prudent, but risk intemperate). Experimental research using the results of E&S and those of the present paper could test whether these two types are the same for risk and for ambiguity.

6.2 Ambiguity prudence and the wide variety of ambiguity models

In section 4, ambiguity prudence was discussed for several widely used (specifications of) ambiguity models, but many other models have been proposed including prospect theory (Tversky and Kahneman, 1992), vector expected utility (Siniscalchi, 2009), second-order subjective expected utility (Seo 2009), uncertainty averse preferences (Cerrei-Vioglio et al., 2011), mean-dispersion preferences (Grant and Polak, 2013), and expected uncertain utility (Gul and Pesendorfer, 2014) as well as the models of Gajdos et al. (2008), Chateauneuf and Faro (2009), and Ergin and Gul (2009). These models have typically been justified by their ability to accommodate ambiguity aversion and by some additional practical or theoretical arguments (smoothness, tractability, relation with financial application, relation with psychological theories). It is clear that we now need behavioural criteria to select among models and ambiguity prudence can help in three different ways.

First, the definition of ambiguity prudence offers a test of ambiguity models. If people were ambiguity neutral at order 3, it would falsify the models discussed in section 4 because they predict strict ambiguity prudence for the usual functional forms and parameters elicited in empirical works or used in theoretical works. If ambiguity prudence is prevailing, (specifications of) models predicting it should be favoured. The definition and example provided in this paper for ambiguity prudence can shed new light on the descriptive value of ambiguity models. Second, if one has doubts about the normative relevance of ambiguity prudence (e.g., thinking that one should be indifferent in the example of ambiguity prudence provided in section 3), then the results of this paper show which models, making such behaviour systematic, should be avoided for normative purposes. Third, we saw how ambiguity prudence plays a critical role in precautionary saving and, above all, in precautionary prevention. Researchers working on prevention behaviour may want to ensure that the ambiguity model they choose allows for precautionary prevention (and can for instance use the smooth model with ambiguity prudence).

6.3 Prudence, prevention, and precautionary behaviour

Risk prudence is tightly linked to the literature on precautionary behaviour, starting with precautionary saving (Kimball, 1990). More recently, Eeckhoudt and Gollier (2005)

and Menegatti (2009) studied the impact of risk prudence on prevention compared with the prevention effort of a risk neutral agent. In a one-period model, Eeckhoudt and Gollier (2005) showed that under some assumptions, risk prudence can reduce prevention. Menegatti (2009) introduced a two-period model in which the effect of prudence is reversed (under the same assumptions as Eeckhoudt and Gollier's). The application presented in section 5.2 highlighted that in the absence of ambiguity, a concave ambiguity function φ may reduce prevention if the second-period expected utility is higher than the first-period utility. When ambiguity is introduced, only ambiguity prudent agents increase their prevention effort (precautionary prevention). The result is comparable to that of Eeckhoudt et al. (2012), who showed that an independent background risk increases the effort level in the presence of risk prudence.

The literature on prevention started with Ehrlich and Becker (1972), who introduced the concepts of self-protection (providing an effort to reduce the probability of a loss as defined in section 5.2) and self-insurance (providing an effort to decrease the magnitude of the loss). Alary et al. (2013) and Berger (2015) studied the impact of ambiguity on insurance, self-insurance, and self-protection. Their settings require stronger conditions than ambiguity prudence to find that future ambiguity increases the optimal effort level. It can be compared to the results of Berger (2014), Gierlinger and Gollier (2014), and Osaki and Schlesinger (2014) on precautionary saving (the increase in saving due to future risk or ambiguity). In settings based on Klibanoff et al.'s (2005) smooth model, they showed that, under some assumptions about the type of ambiguity considered, decreasing absolute ambiguity aversion (DAAA) increased precautionary savings. Ambiguity prudence is not sufficient for future ambiguity to increase saving (Osaki and Schlesinger, 2014). In the online appendix, I provide a simple model of precautionary saving under ambiguity in which ambiguity prudence is sufficient. The formulation of the utility appears to be key in explaining the different results. In the model of precautionary saving in the online appendix and in the prevention model of section 5.2, the utility of each period is expressed in φ -unit, while the aforementioned papers expressed the utility in u -units.

Gierlinger and Gollier (2014) also studied precautionary savings with multiplier preferences and maxmin expected utility. They found that ambiguity always increases savings through a "pessimism effect" when the consumer uses multiplier preferences, but it need not be the case for maxmin expected utility (unless additional conditions on the set of

priors and the utility function u are assumed).

Berger (2014) referred to DAAA as ambiguity prudence. There are two reasons why one might prefer to retain the term ambiguity prudence for the definition proposed in this paper. First, the definition provided in the present paper is independent of the decision model and economic problem discussed. Similarly, the main contribution of E&S was to provide such model-free definitions of higher-order risk preferences. Second, the definition of ambiguity prudence in the present paper directly follows the definition of risk prudence. It is therefore natural to refer to it as ambiguity prudence.

Guerdjikova and Scubba (2015) analysed whether ambiguity-averse agents can survive in a market in which there also are expected utility agents. Ambiguity-averse agents act as expected utility agents would if they had wrong beliefs, which would lead them to behave sub-optimally and be excluded from the market. Guerdjikova and Scubba (2015) showed that agents using the smooth model with φ concave can survive if they satisfy DAAA and if their ambiguity prudence is especially strong ($-\frac{\varphi'''}{\varphi''} \geq -2\frac{\varphi''}{\varphi'}$). This result highlights the need to study higher order ambiguity preferences.

6.4 Extension of ambiguity prudence

We have established how the concept of risk prudence can be extended to ambiguity. The employed recipe can also be applied to higher orders of risk preferences to derive higher orders of ambiguity preferences. E&S defined risk temperance as avoiding bearing two risks simultaneously; we can thus define ambiguity temperance as avoiding bearing two ambiguities simultaneously. The next paragraph illustrates this definition.

Consider a deck of 4 cards, which are either all black or all red (first source of ambiguity). One of the cards is a King, and another a Queen. The two others are undetermined. One of them is a Jack or an Ace (second source of ambiguity), and the other is either a 9 or a 10 (third source of ambiguity). A decision maker can choose between two options, both giving the same prize. With Option A, she wins if the card is a red Queen, a red King, a black Jack, or a black 10. Option B yields the prize if the card is a red Queen, a red Jack, a black King, or a black 10. In other words, Option A combines the Jack-related ambiguity with the 10-related ambiguity in the black-card case while Option B splits them between the red-card and the black-card cases. The ambiguity temperate option is thus the latter.

The online appendix expands all definitions from risk to ambiguity for all orders n . Hansen and Sargent's (2001) multiplier preferences are found to imply all orders of ambiguity apportionment; if the state space is rich enough, ambiguity apportionment of order n determines the sign of the n^{th} derivative of the ambiguity function φ of the two-stage expected utility model.

For ambiguity aversion, we only considered binary spreads $\pm\varepsilon$. However, Ellsberg's (1961) original paradox entails a series of such spreads to show ambiguity aversion. His paradox is based on a known urn that has 50 red and 50 black balls and an unknown urn that can have any 100 red and black balls. Betting on red (or on black), the unknown urn implies a series of binary spreads $\pm 50\%, \dots, \pm 1\%$ in terms of the probability of winning with respect to the known urn. In the online appendix, the definition of ambiguity aversion (and all other definitions) are adapted to allow for any series of binary spreads (and not only one spread as in the main text). The online appendix also provides a way to adapt the definition to some deviations from expected utility under risk.

7 Conclusion

To date, the literature on prudence, temperance, and higher-order preferences has only considered risk (i.e., situations in which probabilities are known). The contribution of this paper was to propose a definition of ambiguity prudence and to analyse it for several ambiguity models. To derive concepts for ambiguity from the concepts defined for risk, the key element was to replace money with probabilities. For risk, prudence means avoiding the combination of a loss with variations of outcomes. For ambiguity, prudence means avoiding the combination of a probability loss (fewer chances of obtaining a desirable outcome) with ambiguity (variations of the probability of obtaining this desirable outcome). The definitions of all orders of ambiguity preferences (and their implications for several ambiguity models) can be found in the online appendix.

Finally, this paper showed that ambiguity prudence can play a role similar to that of risk prudence in determining optimal precautionary behaviour (saving and prevention). It was also noted that going beyond ambiguity aversion may help to better understand people's actual financial behaviour, as shown by Dimmock et al. (2015). Many studies tend to restrict themselves to second order ambiguity preferences (ambiguity aversion).

This paper proposed going beyond ambiguity aversion to enrich our understanding of economic and financial behaviour under ambiguity.

A Appendix: Proof of Theorems 1 to 6 and of Corollary 1

The appendix contains the proofs of all theorems of section 4.

A.1 Proof of Theorem 1 (second order expected utility)

Proof. Assume that, for a couple of choices given by Table 2 in definition 1, the right-hand acts are preferred to the left-hand acts. We can define u such that $Eu(\omega + x) = 1$ and $Eu(\omega + y) = 0$. Under second order expected utility, strictly preferring twice the right-hand act implies $2\varphi(p) < \varphi(p + \varepsilon) + \varphi(p - \varepsilon)$, which implies that φ could not be concave. Definition 1 is thus necessary for $\varphi'' \leq 0$.

First define $E = E_1 \cup E_2$, where E_1 and E_2 come from Richness Assumption 1 (which implies $P(E) = \frac{1}{2}$). Assume that φ is not concave, i.e., it is strictly convex on an interval $[p - \varepsilon, p + \varepsilon]$, when u is scaled such that $Eu(\omega + x) = 1$ and $Eu(\omega + y) = 0$ for some x, y , and ω . Hence, $2\varphi(p) < \varphi(p + \varepsilon) + \varphi(p - \varepsilon)$. This inequality implies that the preferences over the acts of Table 2 built from x, y, p, ε , and E as defined in this paragraph will contradict definition 1.

Assume that, for a quadruple of choices given by Table 3 in definition 4, the right-hand acts are preferred to the left-hand acts. We can define u such that $Eu(\omega + x) = 1$ and $Eu(\omega + y) = 0$. Under second order expected utility, strictly preferring the right-hand acts implies

$2\varphi(p - \kappa) + \varphi(p - \varepsilon) + \varphi(p + \varepsilon) < 2\varphi(p) + \varphi(p - \kappa - \varepsilon) + \varphi(p - \kappa + \varepsilon)$. The results of E&S imply that φ''' cannot be positive. Hence, ambiguity prudence is necessary for $\varphi''' \geq 0$.

First consider E_1, E_2, E_3 , and E_4 from Richness Assumption 1 (which implies $P(E_i) = \frac{1}{4}$ for $i \in \{1, 2, 3, 4\}$). Assume that $\varphi''' < 0$ on an interval $[p - \varepsilon - \kappa, p + \varepsilon]$, when u is scaled such that $Eu(\omega + x) = 1$ and $Eu(\omega + y) = 0$ for some x, y , and ω . From E&S, we know $2\varphi(p - \kappa) + \varphi(p - \varepsilon) + \varphi(p + \varepsilon) < 2\varphi(p) + \varphi(p - \kappa - \varepsilon) + \varphi(p - \kappa + \varepsilon)$. Consequently, the preferences over the acts of Table 3 built from $x, y, p, \kappa, \varepsilon, E_1, E_2, E_3$, and E_4 as defined in this paragraph will contradict definition 4. \square

A.2 Proof of Corollary 1 (expected utility)

Under expected utility, the φ function of second-order expected utility is linear. It is straightforward that only subjective probabilities assigned to events E and E^c in Definition

1 (to E_1, E_2, E_3 , and E_4 in Definition 4) can generate a preference between the left-hand acts and the right-hand acts of Table 1 (Table 3) because all acts provide the same expected utility if all events are equally likely. Hence, permutations will reverse any strict preference (preserve indifference) and therefore, ambiguity neutrality of order 2 (3) will hold.

A.3 Proof of Theorem 2 (smooth model)

Proof. Assume that for a couple of choices given by Table 2 in definition 1, the right-hand acts are preferred to the left-hand acts. We can define u such that $Eu(\omega + x) = 1$ and $Eu(\omega + y) = 0$. Under the smooth model, strictly preferring twice the right-hand act implies

$2\varphi(p) < \int_{\Delta(S)} \mu(P) [\varphi(p + (2P(E) - 1)\varepsilon) + \varphi(p - (2P(E) - 1)\varepsilon)] d\mu(P)$, which implies that φ could not be concave. Definition 1 is thus necessary for $\varphi'' \leq 0$.

Assume that φ is not concave, i.e., it is strictly convex on an interval $[p - \varepsilon, p + \varepsilon]$, when u is scaled such that $Eu(\omega + x) = 1$ and $Eu(\omega + y) = 0$ for some x, y , and ω . Hence, with $\eta \in [-1, 1]$ $2\varphi(p) < \varphi(p + \eta \times \varepsilon) + \varphi(p - \eta \times \varepsilon)$.

Replacing η by $(2P(E) - 1)$ and averaging with μ over $\Delta(S)$

$2\varphi(p) < \int_{\Delta(S)} \mu(P) [\varphi(p + (2P(E) - 1)\varepsilon) + \varphi(p - (2P(E) - 1)\varepsilon)] d\mu(P)$. Let $E = E_1^1$ from Richness Assumption 2. This implies that

$\int_{\Delta(S)} \mu(P) \varphi(p + (2P(E) - 1)\varepsilon) d\mu(P) = \int_{\Delta(S)} \mu(P) \varphi(p - (2P(E) - 1)\varepsilon) d\mu(P)$ and therefore both

$$\varphi(p) < \int_{\Delta(S)} \mu(P) \varphi(p + (2P(E) - 1)\varepsilon) d\mu(P)$$

and

$\varphi(p) < \int_{\Delta(S)} \mu(P) \varphi(p - (2P(E) - 1)\varepsilon) d\mu(P)$. Consequently, the preferences over the acts of Table 2 built from x, y, p , and ε as defined in this paragraph and with $E = E_1^1$ from Richness Assumption 2 will contradict definition 1.

Assume that for a quadruple of choices given by Table 3 in definition 4, the right-hand acts are preferred to the left-hand acts. We can define u such that $Eu(\omega + x) = 1$ and $Eu(\omega + y) = 0$. Under the smooth model, strictly preferring the right-hand acts implies

$$\begin{aligned} & \int_{\Delta(S)} \mu(P) \sum_{i=1}^2 \sum_{j=1}^2 \varphi(p - P(E_i^1)\kappa + (1 - P(E_i^1))(2P(E_j^2) - 1)\varepsilon) d\mu(P) \\ & < \int_{\Delta(S)} \mu(P) \sum_{i=1}^2 \sum_{j=1}^2 \varphi(p + (1 - P(E_i^1))(-\kappa + (2P(E_j^2) - 1)\varepsilon)) d\mu(P). \end{aligned}$$

Consider any P . Through the two summations, φ is applied to four values, and these four values are the combination of a loss $-(1 - P(E_1^1))\kappa$ or $-P(E_1^1)\kappa$ and a variation $\pm(1 - P(E_1^1))(2P(E_1^2) - 1)\varepsilon$ or $\pm P(E_1^1)(2P(E_1^2) - 1)\varepsilon$. If $P(E_1^1) \leq \frac{1}{2}$, then $-(1 - P(E_1^1))\kappa$ first-order stochastically dominates $-P(E_1^1)\kappa$ and $\pm(1 - P(E_1^1))(2P(E_1^2) - 1)\varepsilon$ (with equal probability) second-order stochastically dominates $\pm P(E_1^1)(2P(E_1^2) - 1)\varepsilon$. If $P(E_1^1) \geq \frac{1}{2}$, these dominance relations are reversed. Theorem 3 of Eeckhoudt et al. (2009) implies that a random variable assigning equal probability to $\{-P(E_i^1)\kappa + (1 - P(E_i^1))(2P(E_j^2) - 1)\varepsilon\}$

with $i, j \in \{1, 2\}$ third-order stochastically dominates another variable assigning equal probability to $\{(1 - P(E_i^1))(-\kappa + (2P(E_j^2) - 1)\varepsilon)\}$ with $i, j \in \{1, 2\}$. Theorem 2 of Eeckhoudt et al. (2009) implies that φ''' cannot be positive. Hence, ambiguity prudence is necessary for $\varphi''' \geq 0$.

Assume that $\varphi''' < 0$ on an interval $[p - \varepsilon - \kappa, p + \varepsilon]$, when u is scaled such that $Eu(\omega + x) = 1$ and $Eu(\omega + y) = 0$ for some x, y , and ω . Hence, with $\eta \in [-1, 1], \psi \in [0, 1]$, Theorems 2 and 3 of Eeckhoudt et al. (2009) imply

$$\begin{aligned} & \varphi(p - \psi\kappa + (1 - \psi)\eta\varepsilon) + \varphi(p - \psi\kappa - (1 - \psi)\eta\varepsilon) \\ & \quad + \varphi(p - (1 - \psi)\kappa + \psi\eta\varepsilon) + \varphi(p - (1 - \psi)\kappa - \psi\eta\varepsilon) \\ & < \varphi(p + (1 - \psi)(-\kappa + \eta\varepsilon)) + \varphi(p + (1 - \psi)(-\kappa - \eta\varepsilon)) \\ & \quad + \varphi(p + \psi(-\kappa + \eta\varepsilon)) + \varphi(p + \psi(-\kappa - \eta\varepsilon)) \quad (8) \end{aligned}$$

Replacing η by $(2P(E_1^2) - 1)$, ψ by $P(E_1^1)$ and averaging with μ over $\Delta(S)$

$$\begin{aligned} & \int_{\Delta(S)} \mu(P) \sum_{i=1}^2 \sum_{j=1}^2 \varphi(p - P(E_i^1)\kappa + (1 - P(E_i^1))(2P(E_j^2) - 1)\varepsilon) d\mu(P) \\ & < \int_{\Delta(S)} \mu(P) \sum_{i=1}^2 \sum_{j=1}^2 \varphi(p + (1 - P(E_i^1))(-\kappa + (2P(E_j^2) - 1)\varepsilon)) d\mu(P). \end{aligned}$$

Let E_1^1 and E_1^2 be as in Richness Assumption 2. It implies that for all $i, j \in \{1, 2\}$,

$$\int_{\Delta(S)} \mu(P) \varphi(p - P(E_i^1)\kappa + (1 - P(E_i^1))(2P(E_j^2) - 1)\varepsilon) d\mu(P) = c$$

and

$$\int_{\Delta(S)} \mu(P) \varphi(p + (1 - P(E_i^1))(-\kappa + (2P(E_j^2) - 1)\varepsilon)) d\mu(P) = k$$

for some c and k . Furthermore, $c < k$. Consequently, the preferences over the acts of Table 3 built from x, y, p, κ , and ε as defined in this paragraph and with E_1^1 and E_1^2 from Richness Assumption 2 will contradict definition 4. \square

A.4 Proof of Theorem 3 (multiple-prior models)

Proof. If preferences are represented by α -maxmin with $C = \{P : P = (1 - \epsilon)Q + \epsilon T, T \in \Delta(S)\}$ for a given probability measure Q and a contamination $0 \leq \epsilon \leq 1$, then

$$\begin{aligned} & \alpha \min_{P \in C} \left(\int_S Eu(\omega + f(s)) dP(s) \right) + (1 - \alpha) \max_{P \in C} \left(\int_S Eu(\omega + f(s)) dP(s) \right) \\ & = (1 - \epsilon) Eu(\omega + f(s)) dQ(s) \\ & \quad + \epsilon \alpha \min_{T \in \Delta(S)} \left(\int_S Eu(\omega + f(s)) dT(s) \right) + \epsilon(1 - \alpha) \max_{T \in \Delta(S)} \left(\int_S Eu(\omega + f(s)) dT(s) \right) \\ & = (1 - \epsilon) Eu(\omega + f(s)) dQ(s) + \epsilon \alpha \min_{s \in S} (Eu(\omega + f(s))) + \epsilon(1 - \alpha) \max_{s \in S} (Eu(\omega + f(s))). \end{aligned}$$

Hence, if $Q(E) > 0$ for all $E \in \Sigma \setminus \{S\}$, it can be written as a neo-additive Choquet expected utility with $a = \epsilon$ and $b = a(2\alpha - 1)$. Theorem 3 follows from Theorem 6. The restriction about Q is only necessary to identify the best and the worst expected utilities in neo-additive Choquet expected utility. In the multiple prior models with ϵ -contamination, the best and worst expected utilities are taken over all $s \in S$ and therefore the restriction about Q is not necessary. \square

A.5 Proof of Theorem 4 (multiplier preferences)

Proof. Multiplier preferences are ordinally equivalent to second-order expected utility with φ_θ as defined in Eq.1. From the derivatives of the exponential function, we conclude that $\text{sgn}(\varphi_\theta^{(n)}) = (-1)^{n+1}$ for all n . From Theorem 1, we know it implies ambiguity aversion and ambiguity prudence. Moreover, Richness Assumption 1 ensures the existence of 4 events whose baseline probability Q is $\frac{1}{4}$. For such events, and if $\theta < \infty$ (thus excluding expected utility), the left-hand acts of Tables 1 and 3 will be strictly preferred to their respective right-hand acts as soon as $x \succ y$, $p \in (0, 1)$, $\varepsilon > 0$ and $\kappa > 0$. Hence, under Richness Assumption 1, $\theta < \infty$ implies strict ambiguity aversion and strict ambiguity prudence. \square

A.6 Proof of Theorem 5 (Choquet expected utility)

Proof. Assume that for a couple of choices given by Table 2 in definition 1, the right-hand acts are preferred to the left-hand acts. We can define u such that $Eu(\omega + x) = 1$ and $Eu(\omega + y) = 0$. Under Choquet expected utility, strictly preferring twice the right-hand act implies $1 < W(E) + W(E^c)$ and thus violates binary superadditivity. Hence, Definition 1 is necessary for binary superadditivity.

Assume that for a quadruple of choices given by Table 3 in definition 4, the right-hand acts are preferred to the left-hand acts. We can define u such that $Eu(\omega + x) = 1$ and $Eu(\omega + y) = 0$. We assume that there exists $\eta > 0$ such that $\eta < W(E) < 1 - \eta$ $\forall E \in \Sigma \setminus \{\emptyset, S\}$. Under Choquet expected utility, strictly preferring the right-hand act in all four choices of Table 3 implies $4 < 2 \sum_{i=1}^4 W(E_i^c) - 2 \sum_{i=1}^4 W(E)$. Upper subadditivity implies $4 - \sum_{i=1}^4 W(E_i^c) \geq 2W(E_1 \cup E_2) + 2W(E_3 \cup E_4) - \sum_{i=1}^4 W(E_i)$, and lower subadditivity implies $2W(E_1 \cup E_2) + 2W(E_3 \cup E_4) + \sum_{i=1}^4 W(E_i) \geq \sum_{i=1}^4 W(E_i^c)$. Hence, upper and lower subadditivity implies $4 \geq 2 \sum_{i=1}^4 W(E_i^c) - 2 \sum_{i=1}^4 W(E)$ and therefore ambiguity prudence. \square

A.7 Proof of Theorem 6 (neo-additive Choquet expected utility)

Proof. First note that $b > 0$ implies that W is binary superadditive. From Theorem 5, we immediately derive that it implies ambiguity aversion. If $b < 0$, then $1 < W(E) + W(E^c)$ for all E such that $P(E) > 0$. We can pick lotteries x and y such that $x \succ y$ and define u such that $Eu(\omega + x) = 1$ and $Eu(\omega + y) = 0$. Let us also consider event $E = E_1 \cup E_2$ where E_1 and E_2 come from Richness Assumption 1 (which implies $W(E) = W(E^c) > \frac{1}{2}$). Using these events and $\varepsilon > 0$, the right-hand acts of Table 1 will be strictly preferred to the left-hand acts, and ambiguity aversion will not hold. Hence, $b \geq 0$ is equivalent to ambiguity aversion under Richness Assumption 1.

If W is neo-additive, then it satisfies upper and lower subadditivity. Assume $P(E) > 0$

$\forall E \in \Sigma \setminus \emptyset$. Then we can derive from Theorem 5 that prudence holds. Now pick lotteries x and y such that $x \succ y$ and define u such that $Eu(\omega + x) = 1$ and $Eu(\omega + y) = 0$. Take $\kappa > \varepsilon > 0$. Consider event E_1, E_2, E_3 , and E_4 from Richness Assumption 1 (which implies $P(E_i) = \frac{1}{4}$ for $i \in \{1, 2, 3, 4\}$). Using these events, the left-hand acts and the right-hand acts of Table 3 will have the same overall expected utility. Hence, only the worst and the best expected utilities will matter. In the left-hand acts, the extreme expected utilities are $p - \kappa$ and $p + \varepsilon$, and the extreme right-hand acts are $p - \kappa - \varepsilon$ and p . The worst and the best expected utilities are strictly higher in the left-hand acts than in the right-hand acts, which means that the former will be strictly preferred to the latter if $a > 0$. Hence, under Richness Assumption 1, $a > 0$ implies strict ambiguity prudence. \square

References

- Abdellaoui, M., Baillon, A., Placido, L., and Wakker, P. P. (2011). The rich domain of uncertainty: Source functions and their experimental implementation. *The American Economic Review*, 101(2):695–723.
- Alary, D., Gollier, C., and Treich, N. (2013). The effect of ambiguity aversion on insurance and self-protection. *The Economic Journal*, 123(573):1188–1202.
- Anscombe, F. J. and Aumann, R. J. (1963). A definition of subjective probability. *The Annals of Mathematical Statistics*, 34(1):pp. 199–205.
- Arditti, F. D. (1967). Risk and the required return on equity. *The Journal of Finance*, 22(1):19–36.
- Baillon, A., Cabantous, L., and Wakker, P. (2012). Aggregating imprecise or conflicting beliefs: An experimental investigation using modern ambiguity theories. *Journal of Risk and Uncertainty*, 44:115–147.
- Berger, L. (2014). Precautionary saving and the notion of ambiguity prudence. *Economics letters*, 123(2):248–251.
- Berger, L. (2015). The impact of ambiguity prudence on insurance and prevention.
- Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., and Montrucchio, L. (2011). Uncertainty averse preferences. *Journal of Economic Theory*, 146(4):1275–1330.
- Chateauneuf, A., Eichberger, J., and Grant, S. (2007). Choice under uncertainty with the best and worst in mind: Neo-additive capacities. *Journal of Economic Theory*, 137(1):538–567.
- Chateauneuf, A. and Faro, J. H. (2009). Ambiguity through confidence functions. *Journal of Mathematical Economics*, 45(9-10):535–558.
- Chateauneuf, A. and Tallon, J.-M. (2002). Diversification, convex preferences and non-empty core in the Choquet expected utility model. *Economic Theory*, 19(3):509–523.
- Crainich, D., Eeckhoudt, L., and Trannoy, A. P. (2013). Even (mixed) risk lovers are prudent. *The American Economic Review*, 103(4):1536–1537.

- Deck, C. and Schlesinger, H. (2010). Exploring higher order risk effects. *The Review of Economic Studies*, 77(4):1403–1420.
- Deck, C. and Schlesinger, H. (2014). Consistency of higher order risk preferences. *Econometrica*, 82(5):1913–1943.
- Dillenberger, D. and Segal, U. (2013). Skewed noise.
- Dimmock, S. G., Kouwenberg, R., , and Wakker, P. P. Ambiguity attitudes in a large representative sample. *mimeo*.
- Ebert, S. (2012). Mixed risk aversion and the apportionment of skewed risks. *mimeo*.
- Ebert, S. and Wiesen, D. (2011). Testing for prudence and skewness seeking. *Management Science*, 57(7):1334–1349.
- Eeckhoudt, L. and Gollier, C. (2005). The impact of prudence on optimal prevention. *Economic Theory*, 26(4):989–994.
- Eeckhoudt, L., Huang, R. J., and Tzeng, L. Y. (2012). Precautionary effort: a new look. *Journal of Risk and Insurance*, 79(2):585–590.
- Eeckhoudt, L. and Schlesinger, H. (2006). Putting risk in its proper place. *American Economic Review*, 96(2):280–289.
- Eeckhoudt, L., Schlesinger, H., and Tsetlin, I. (2009). Apportioning of risks via stochastic dominance. *Journal of Economic Theory*, 144(3):994–1003.
- Ehrlich, I. and Becker, G. S. (1972). Market insurance, self-insurance, and self-protection. *The Journal of Political Economy*, pages 623–648.
- Ellsberg, D. (1961). Risk, ambiguity and the Savage axioms. *Quarterly Journal of Economics*, 75:643–669.
- Epstein, L. G. (1999). A definition of uncertainty aversion. *The Review of Economic Studies*, 66(3):579–608.
- Epstein, L. G. and Wang, T. (1994). Intertemporal asset pricing under knightian uncertainty. *Econometrica*, pages 283–322.
- Ergin, H. and Gul, F. (2009). A theory of subjective compound lotteries. *Journal of Economic Theory*, 144(3):899–929.
- Gajdos, T., Hayashi, T., Tallon, J.-M., and Vergnaud, J.-C. (2008). Attitude toward imprecise information. *Journal of Economic Theory*, 140(1):27–65.
- Ghirardato, P., Maccheroni, F., and Marinacci, M. (2004). Differentiating ambiguity and ambiguity attitude. *Journal of Economic Theory*, 118(2):133–173.
- Ghirardato, P. and Marinacci, M. (2002). Ambiguity made precise: A comparative foundation. *Journal of Economic Theory*, 102(2):251–289.
- Gierlinger, J. and Gollier, C. (2014). Saving for an ambiguous future. *mimeo*.
- Gilboa, I. and Schmeidler, D. (1989). Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18(2):141–153.

- Grant, S. and Polak, B. (2013). Mean-dispersion preferences and constant absolute uncertainty aversion. *Journal of Economic Theory*, 148(4):1361–1398.
- Guerdjikova, A. and Sciubba, E. (2015). Survival with ambiguity. *Journal of Economic Theory*.
- Gul, F. and Pesendorfer, W. (2014). Expected uncertain utility theory. *Econometrica*, 82(1):1–39.
- Halevy, Y. (2007). Ellsberg revisited: An experimental study. *Econometrica*, 75(2):503–536.
- Hansen, L. P. and Sargent, T. J. (2001). Robust control and model uncertainty. *American Economic Review*, 91(2):60–66.
- Huang, Y.-C., Tzeng, L. Y., and Zhao, L. (2015). Comparative ambiguity aversion and downside ambiguity aversion. *Insurance: Mathematics and Economics*, 62:257–269.
- Kimball, M. S. (1990). Precautionary saving in the small and in the large. *Econometrica*, 58(1):pp. 53–73.
- Kimball, M. S. (1992). Precautionary motives for holding assets. In Newman, P., Milgate, M., and Falwell, J., editors, *The New Palgrave Dictionary of Money and Finance*. Macmillan, London.
- Klibanoff, P., Marinacci, M., and Mukerji, S. (2005). A smooth model of decision making under ambiguity. *Econometrica*, 73(6):1849–1892.
- Knight, F. H. (1921). *Risk, Uncertainty, and Profit*. Houghton Mifflin, New York.
- Kraus, A. and Litzenberger, R. H. (1976). Skewness preference and the valuation of risk asset. *The Journal of Finance*, 31(4):1085–1100.
- Lajeri-Chaherli, F. (2004). Proper prudence, standard prudence and precautionary vulnerability. *Economics Letters*, 82(1):29–34.
- Leland, H. E. (1968). Saving and uncertainty: The precautionary demand for saving. *The Quarterly Journal of Economics*, 82(3):465–473.
- Maafi, H. (2011). Preference reversals under ambiguity. *Management Science*, 57(11):2054–2066.
- Maccheroni, F., Marinacci, M., and Rustichini, A. (2006). Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, 74(6):1447–1498.
- Maier, J. and Ruger, M. (2012). Experimental evidence on higher order risk preferences with real monetary losses. *working paper*.
- Menegatti, M. (2009). Optimal prevention and prudence in a two-period model. *Mathematical Social Sciences*, 58(3):393–397.
- Neilson, W. (2010). A simplified axiomatic approach to ambiguity aversion. *Journal of Risk and Uncertainty*, 41:113–124.
- Noussair, C. N., Trautmann, S. T., and van de Kuilen, G. (2012). Higher order risk attitudes, demographics, and financial decisions. *mimeo*.
- Osaki, Y. and Schlesinger, H. (2014). Precautionary saving and ambiguity. *mimeo*.
- Sandmo, A. (1970). The effect of uncertainty on saving decisions. *The Review of Economic Studies*, 37(3):pp. 353–360.

- Savage, L. J. (1954). *The Foundations of Statistics*. 2nd ed 1972, Dover Publications, Wiley, New York.
- Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica*, 57:571–587.
- Seo, K. (2009). Ambiguity and second-order belief. *Econometrica*, 77(5):1575–1605.
- Siniscalchi, M. (2009). Vector expected utility and attitudes toward variation. *Econometrica*, 77(3):801–855.
- Strzalecki, T. (2011). Axiomatic foundations of multiplier preferences. *Econometrica*, 79(1):47–73.
- Tarazona-Gomez, M. (2004). Are individuals prudent? an experimental approach using lottery choices. *mimeo*.
- Trautmann, S. T. and van de Kuilen, G. (forthcoming). Attitudes towards ambiguity. In Keren, G. and Wu, G., editors, *Blackwell Handbook of Judgment and Decision Making*, chapter 11.
- Tversky, A. and Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, 5:297–323.
- Tversky, A. and Wakker, P. (1995). Risk attitudes and decision weights. *Econometrica*, pages 1255–1280.
- Wakker, P. P. (2010). *Prospect Theory for Risk and Ambiguity*. Cambridge University Press, Cambridge, UK.